

Appendix W

Block Diagram Reduction

W.3 Δ Mason's Rule and the Signal-Flow Graph

A compact alternative notation to the block diagram is given by the **signal-flow graph** introduced by S. J. Mason (1953, 1956). As with the block diagram, the signal-flow graph offers a visual tool for representing the causal relationships between the components of the system. The method consists of characterizing the system by a network of directed branches and associated gains (transfer functions) connected at nodes. Several block diagrams and their corresponding signal-flow graphs are shown in Fig. W.1. The two ways of depicting a system are equivalent, and you can use either diagram to apply Mason's rule (to be defined shortly).

In a signal-flow graph the internal signals in the diagram, such as the common input to several blocks or the output of a summing junction, are called **nodes**. The system input point and the system output point are also nodes; the input node has outgoing branches only, and the output node has incoming branches only. Mason defined a **path** through a block diagram as a sequence of connected blocks, the route passing from one node to another *in the direction of signal flow of the blocks* without including any block more than once. A **forward path** is a path from the input to output such that no node is included more than once. If the nodes are numbered in a convenient order, then a forward path can be identified by the numbers that are included. Any closed path that returns to its starting node without passing through any node more than once is a **loop**, and a path that leads from a given variable back to the same variable is a **loop path**. The **path gain** is the product of component gains (transfer functions) making up the path. Similarly, the **loop gain** is the path gain associated with a loop—that is, the product of gains in a loop. If two paths have a common component, they are said to touch. Notice particularly in this connection that the input and the output of a summing junction are not the same and that the summing junction is a one-way device from its inputs to its output.

Mason's rule relates the graph to the algebra of the simultaneous equations it represents.¹ Consider Fig. W.1(c), where the signal at each node has been given a name and the gains are marked. Then the block diagram (or the signal-flow graph) represents the following system of equations:

$$\begin{aligned} X_1(s) &= X_3(s) + U(s), \\ X_2(s) &= G_1(s)X_1(s) + G_2(s)X_2(s) + G_4(s)X_3(s), \\ Y(s) &= 1X_3(s). \end{aligned}$$

Mason's rule states that the input-output transfer function associated with a signal-flow graph is given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{\Delta} \sum_i G_i \Delta_i,$$

Mason's rule

¹The derivation is based on Cramer's rule for solving linear equations by determinants and is described in Mason's papers.

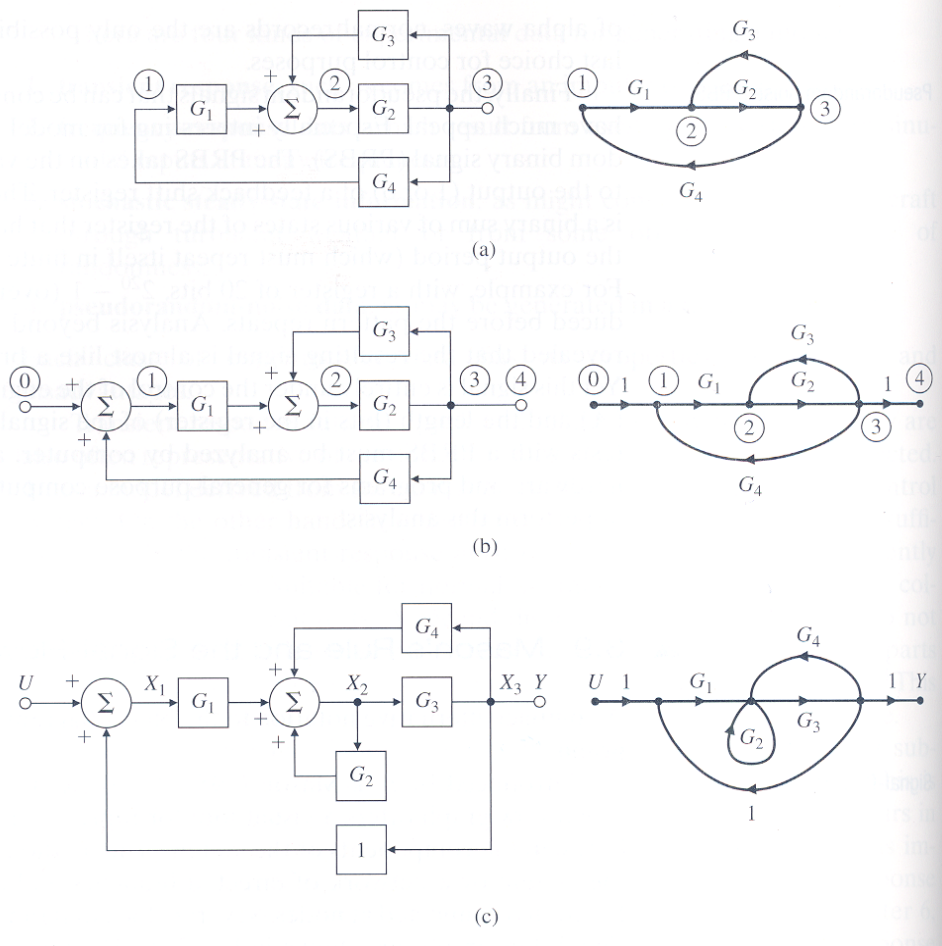


Figure W.1: Block diagrams and corresponding signal flow graphs

where

- G_i = path gain of the i th forward path,
- Δ = the system determinant
 $1 - \sum$ (all individual loop gains) + \sum (gain products of all possible two loops that do not touch) - \sum (gain products of all possible three loops that do not touch) + \dots ,
- Δ_i = i th forward path determinant
 = value of Δ for that part of the block diagram that does *not* touch the i th forward path.

We will now illustrate the use of Mason's rule by several examples.

Example W.1 *Mason's Rule in a Simple System Find the transfer function for the block diagram in Fig. W.2.*

SOLUTION From the block diagram shown in Fig. W.2 we have

| Forward Path | Path Gain |
|--------------|---|
| 1236 | $G_1 = 1 \left(\frac{1}{s}\right) (b_1)(1)$ |
| 12346 | $G_2 = 1 \left(\frac{1}{s}\right) \left(\frac{1}{s}\right) (b_2)(1)$ |
| 123456 | $G_3 = 1 \left(\frac{1}{s}\right) \left(\frac{1}{s}\right) \left(\frac{1}{s}\right) (b_3)(1)$ |
| | <i>Loop Path Gain</i> |
| 232 | $l_1 = -a_1/s$ |
| 2342 | $l_2 = -a_2/s^2$ |
| 23452 | $l_3 = -a_3/s^3$ |

and the determinants are

$$\begin{aligned} \Delta &= 1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3}\right) + 0 \\ \Delta_1 &= 1 - 0 \\ \Delta_2 &= 1 - 0 \\ \Delta_3 &= 1 - 0. \end{aligned}$$

Applying Mason's rule, we find the transfer function to be

$$\begin{aligned} G(s) &= \frac{Y(s)}{U(s)} = \frac{(b_1/s) + (b_2/s^2) + (b_3/s^3)}{1 + (a_1/s) + (a_2/s^2) + (a_3/s^3)} \\ &= \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}. \end{aligned}$$

Mason's rule is particularly useful for more complex systems where there are several loops, some of which do not sum into the same point.

Example W.2 *Mason's Rule in a Complex System Find the transfer function for the system shown in Fig. W.3.*

SOLUTION From the block diagram, we find that

| Forward Path | Path Gain |
|--------------|---|
| 12456 | $G_1 = H_1 H_2 H_3$ |
| 1236 | $G_2 = H_4$ |
| | <i>Loop Path Gain</i> |
| 242 | $l_1 = H_1 H_5$ (does not touch l_3) |

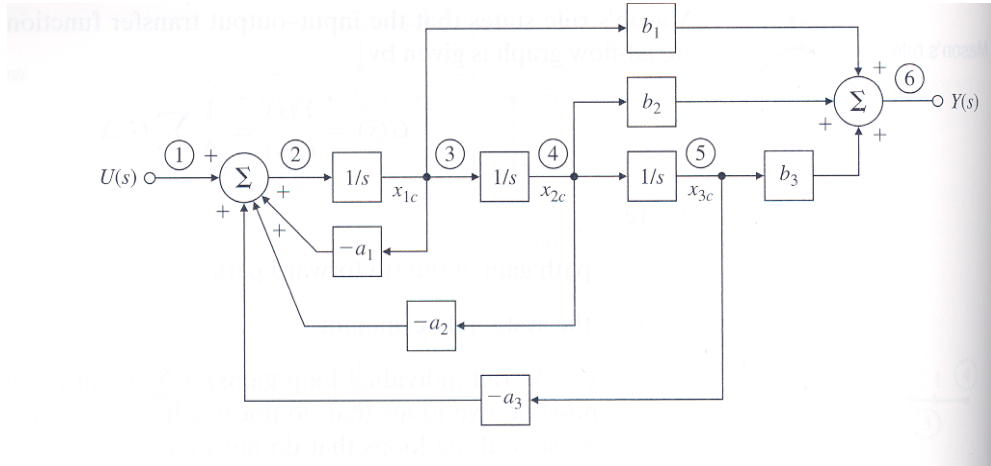


Figure W.2: Block diagram for Example W.1

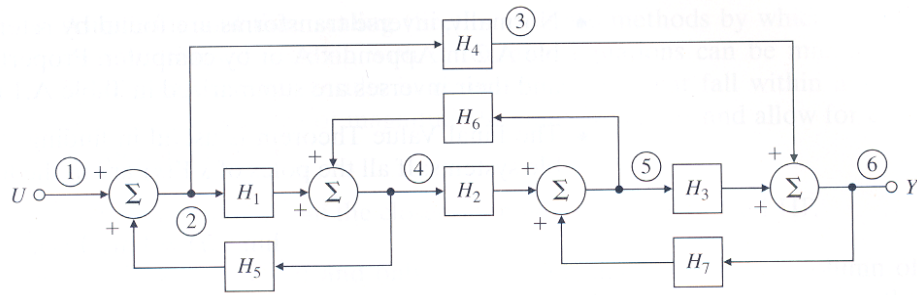


Figure W.3: Block diagram for Example W.2

$$\begin{aligned}
 454 & \quad l_2 = H_2 H_6 \\
 565 & \quad l_3 = H_3 H_7 \text{ (does not touch } l_1) \\
 236542 & \quad l_4 = H_4 H_7 H_6 H_5
 \end{aligned}$$

and the determinants are

$$\begin{aligned}
 \Delta &= 1 - (H_1 H_5 + H_2 H_6 + H_3 H_7 + H_4 H_7 H_6 H_5) + (H_1 H_5 H_3 H_7) \\
 \Delta_1 &= 1 - 0 \\
 \Delta_2 &= 1 - H_2 H_6.
 \end{aligned}$$

Therefore,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{H_1 H_2 H_3 + H_4 - H_4 H_2 H_6}{1 - H_1 H_5 - H_2 H_6 - H_3 H_7 - H_4 H_7 H_6 H_5 + H_1 H_5 H_3 H_7}.$$

Mason's rule is useful for solving relatively complicated block diagrams by hand. It yields the solution in the sense that it provides an explicit input-output relationship for the system represented by the diagram. The advantage compared with path-by-path block-diagram reduction is that it is systematic and algorithmic rather than problem dependent. MATLAB and other control systems computer-aided software allow you to specify a system in terms of individual blocks in an overall

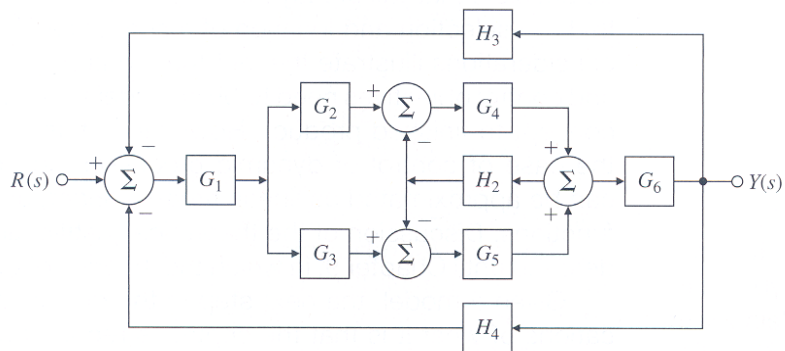


Figure W.4: Block diagram for Problem 2

system, and the software algorithms perform the required block-diagram reduction; therefore, Mason's rule is less important today than in the past. However, there are some derivations that rely on the concepts embodied by the rule, so it still has a role in the control designer's toolbox.

Problems: Mason's Rule and the Signal-Flow Graph

1. Δ Find the transfer functions for the block diagrams in Fig. 3.53, using Mason's rule.
2. Δ Use block-diagram algebra or Mason's rule to determine the transfer function between $R(s)$ and $Y(s)$ in Fig. W.4.