△Appendix W.3.6.3.1 Routh Special Cases

If only the first element in one of the rows is zero, then we can consider a modified equation with one of the coefficients perturbed by $\epsilon > 0$ and applying the test by taking the limit as $\epsilon \to 0$.

EXAMPLE W3.3 *Routh's Test for Special Case I*

Consider the polynomial

$$a(s) = s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9.$$

Determine whether any of the roots are in the RHP.

Solution. In this example, let the coefficient of s^3 be $2 + \epsilon$. The test follows from there. The Routh array is

$$\begin{array}{rlll} s^5: & 1 & 2 & 6 \\ s^4: & 3 & 6 & 9 \\ s^3: & \epsilon & 3 & 0 \\ s^2: & \frac{6\epsilon-9}{\epsilon} & 9 & 0 \\ s: & 3 - \frac{3\epsilon^2}{2\epsilon-3} & 0 & 0 \\ s^0: & 9 & 0. & \end{array}$$

There are two sign changes in the first column of the array, which means there are two poles not in the LHP.[1]

Special case II

Another special[2] case occurs when an entire row of the Routh array is zero. This indicates that there are complex conjugate pairs of roots that are mirror images of each other with respect to the imaginary axis. To apply Routh's test correctly, we follow the ensuing procedure. If the ith row is zero, we form an auxiliary equation from the previous (nonzero) row:

$$a_1(s) = \beta_1 s^{i+1} + \beta_2 s^{i-1} + \beta_3 s^{i-3} + \cdots . \qquad \text{(W3.1)}$$

Here $\{\beta_i\}$ are the coefficients of the $(i+1)$th row in the array. We then replace the ith row by the coefficients of the *derivative* of the auxiliary polynomial and complete the array. However, the roots of the auxiliary polynomial in Eq. (W3.1) are also roots of the characteristic equation, and these must be tested separately.

[1]The actual roots computed with Matlab are at $-2.9043, 0.6567 \pm 1.2881j$, $-0.7046 \pm 0.9929j$.

[2]Special case II.

EXAMPLE W3.4 *Routh Test for Special Case II*

For the polynomial

$$a(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12,$$

determine whether there are any roots on the $j\omega$ axis or in the RHP.

Solution. The Routh array is

$$\begin{array}{rllll}
s^5: & 1 & 11 & 28 & \\
s^4: & 5 & 23 & 12 & \\
s^3: & 6.4 & 25.6 & 0 & \\
s^2: & 3 & 12 & & \\
s: & 0 & 0 & & \leftarrow a_1(s) = 3s^2 + 12 \\
\text{New } s: & 6 & 0 & & \leftarrow \frac{da_1(s)}{ds} = 6s \\
s^0: & 12. & & &
\end{array}$$

There are no sign changes in the first column. Hence all the roots have negative real parts except for a pair on the imaginary axis. We may deduce this as follows: When we replace the zero in the first column by $\epsilon > 0$, there are no sign changes. If we let $\epsilon < 0$, then there are two sign changes. Thus, if $\epsilon = 0$, there are two poles on the imaginary axis, which are the roots of

$$a_1(s) = 3s^2 + 12 = 0,$$

or

$$s = \pm j2.$$

This agrees with the fact that the actual roots are at -3, $\pm 2j$, -1, and -1, as computed using the roots command in Matlab.