Appendix W6.8.1
Time Delay via the Nyquist Diagram

EXAMPLE W6.2 *Nyquist Plot for a System with Time Delay*

Consider the system with

$$KG(s) = \frac{Ke^{-T_d s}}{s},$$

where $T_d = 1$ sec. Determine the range of K for which the system is stable.

Solution. Because the Bode plotting rules do not apply for the phase of a time-delay term, we will use an analytical approach to determine the key features of the frequency response plot. As just discussed, the magnitude of the frequency response of the delay term is unity, and its phase is $-\omega$ radians. The magnitude of the frequency response of the pure integrator is $1/\omega$ with a constant phase of $-\pi/2$. Therefore,

$$\begin{aligned} G(j\omega) &= \frac{1}{\omega} e^{-j(\omega+\pi/2)} \\ &= \frac{1}{\omega}(-\sin\omega - j\cos\omega). \end{aligned} \tag{W6.6}$$

Using Eq. (W6.6) and substituting in different values of ω, we can generate the Nyquist plot, which is the spiral shown in Fig. W6.6.

Let us examine the shape of the spiral in more detail. We pick a Nyquist path with a small detour to the right of the origin. The effect of the pole at the origin is the large arc at infinity with a 180° sweep, as shown in Fig. W6.6. From Eq. (W6.6), for small values of $\omega > 0$, the real part of the frequency response is close to -1 because $\sin\omega \cong \omega$ and $\mathrm{Re}[G(j\omega)] \cong -1$. Similarly, for small values of $\omega > 0$, $\cos\omega \cong 1$ and $\mathrm{Im}[G(j\omega)] \cong -1/\omega$—that is, very large negative values, as shown in Fig. W6.6. To obtain the crossover points on the real axis, we set the imaginary part equal to zero:

$$\frac{\cos\omega}{\omega} = 0. \tag{W6.7}$$

The solution is then

$$\omega_0 = \frac{(2n+1)\pi}{2}, \qquad n = 0, 1, 2, \ldots. \tag{W6.8}$$

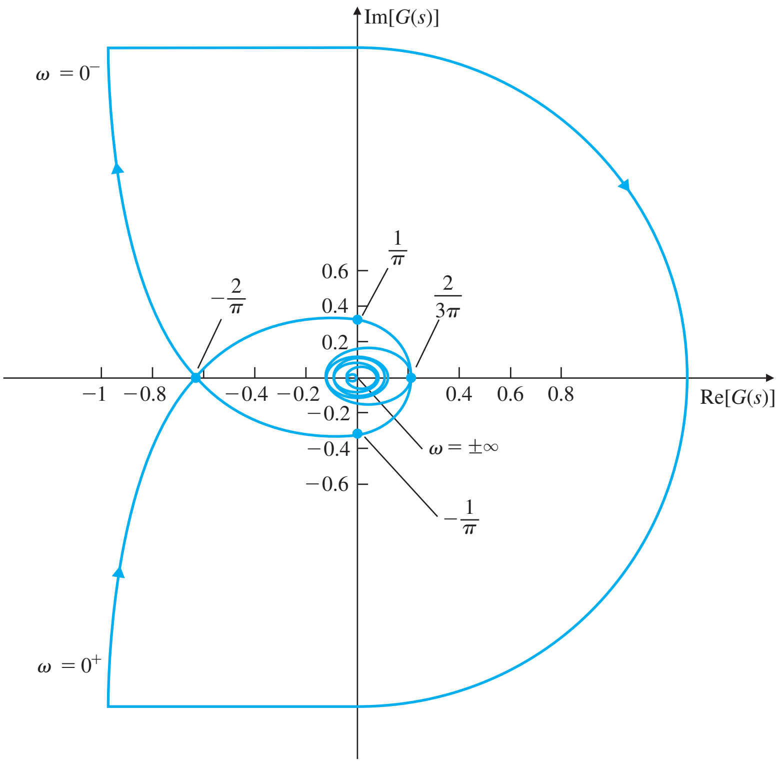

Figure W6.6
Nyquist plot for Example W6.2

After substituting Eq. (W6.8) back into Eq. (W6.6), we find that

$$G(j\omega_0) = \frac{(-1)^n}{(2n+1)}\left(\frac{2}{\pi}\right), \quad n = 0, 1, 2, \ldots .$$

So the first crossover of the negative real axis is at $-2/\pi$, corresponding to $n = 0$. The first crossover of the positive real axis occurs for $n = 1$ and is located at $2/3\pi$. As we can infer from Fig. W6.6, there are an infinite number of other crossings of the real axis. Finally, for $\omega = \infty$, the Nyquist plot converges to the origin. Note that the Nyquist plot for $\omega < 0$ is the mirror image of the one for $\omega > 0$.

The number of poles in the RHP is zero ($P = 0$), so for closed-loop stability, we need $Z = N = 0$. Therefore, the Nyquist plot cannot be allowed to encircle the $-1/K$ point. It will not do so as long as

$$-\frac{1}{K} < -\frac{2}{\pi}, \tag{W6.9}$$

which means that, for stability, we must have $0 < K < \pi/2$.