Appendix W4.1.4.1
The Filtered Case

Thus far, the analysis has been based on the simplest open- and closed-loop structures. A more general case includes a dynamic filter on the input and also dynamics in the sensor. The filtered open-loop structure is shown in Fig. W4.1 having the transfer function $T_{ol} = GD_{ol}F$. In this case, the open-loop controller transfer function has been simply replaced by $D_{ol}F$, and the discussion given for the unfiltered open-loop case is easily applied to this change.

For the filtered feedback case shown in Fig. W4.2, the changes are more significant. In that case, the transform of the system output is given by

$$Y = \frac{GD_{cl}F}{1 + GD_{cl}H}R + \frac{G}{1 + GD_{cl}H}W - \frac{HGD_{cl}}{1 + GD_{cl}H}V. \qquad \text{(W4.1)}$$

As is evident from this equation, the sensor dynamics, H, is part of the loop transfer function and enters into the question of stability with $D_{cl}H$ replacing the D_{cl} of the unity feedback case. In fact, if $F = H$, then with respect to stability, tracking, and regulation, the filtered case is identical to the unity case with $D_{cl}H$ replacing D_{cl}. On the other hand, the filter transfer function F can play the role of the open-loop controller except that here the filter F would be called on to modify the

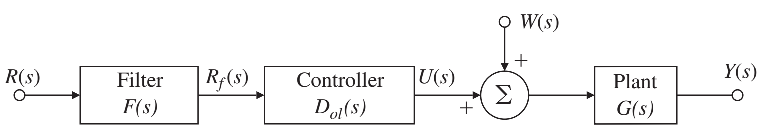

Figure W4.1
Filtered open-loop system

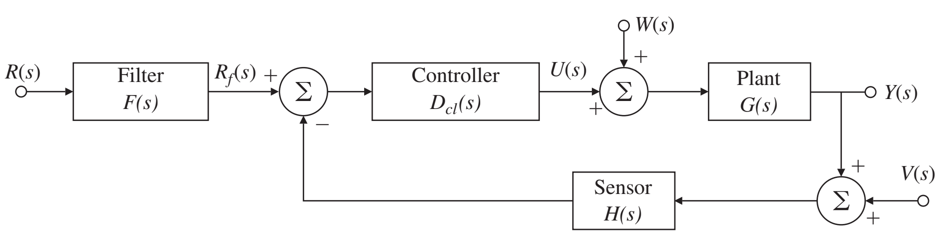

Figure W4.2
Filtered closed-loop. R = reference, U = control, Y = output, V = sensor noise

entire loop transfer function, $\frac{GD_{cl}}{1+GD_{cl}H}$, rather than simply GD_{ol}. Therefore, the filtered closed-loop structure can realize the best properties of both the open-loop and the unity feedback closed-loop cases. The controller, D_{cl}, can be designed to effectively regulate the system for the disturbance W and the sensor noise V, while the filter F is designed to improve the tracking accuracy. If the sensor dynamics H are accessible to the designer, this term also can be designed to improve the response to the sensor noise. The remaining issue is sensitivity.

Using the formula given in Eq. (4.18), with changes in the parameter of interest, we can compute

$$\mathcal{S}_F^{T_{cl}} = 1.0, \tag{W4.2}$$

$$\mathcal{S}_G^{T_{cl}} = \frac{1}{1+GD_{cl}H}, \tag{W4.3}$$

$$\mathcal{S}_H^{T_{cl}} = -\frac{GD_{cl}H}{1+GD_{cl}H}. \tag{W4.4}$$

Of these, the most interesting is the last. Notice with respect to H, the sensitivity approaches unity as the loop gain grows. Therefore it is particularly important that the transfer function of the sensor be not only low in noise but also very stable in gain. Money spent on the sensor is money well spent!

EXAMPLE W4.1

If S is the sensitivity of the filtered feedback system to changes in the plant transfer function, and T is the transfer function from reference to output, compute the sum of $S+T$. Show that $S+T=1$ if $F=H$.

(a) Compute the sensitivity of the filtered feedback system shown in Fig. W4.2 with respect to changes in the plant transfer function, G.
(b) Compute the sensitivity of the filtered feedback system shown in Fig. W4.2 with respect to changes in the controller transfer function, D_{cl}.
(c) Compute the sensitivity of the filtered feedback system shown in Fig. W4.2 with respect to changes in the filter transfer function, F.
(d) Compute the sensitivity of the filtered feedback system shown in Fig. W4.2 with respect to changes in the sensor transfer function, H. If S is the sensitivity of the filtered feedback system to changes in the plant transfer function and T is the transfer function from reference to output, compute the sum of $S+T$. Show that $S+T=1$ if $F=H$.

Solution. To answer the first question, we need the answer to part (a), so let's start there.

(a) Applying the formula for sensitivity of T to changes in G:

$$T = \frac{FD_{cl}G}{1 + D_{cl}GH},$$

$$S = G\frac{1 + D_{cl}GH}{FD_{cl}G}\frac{(1 + D_{cl}GH)FD_{cl} - FD_{cl}G(D_{cl}H)}{(1 + D_{cl}GH)^2}$$

$$= \frac{1}{1 + D_{cl}GH}.$$

Now

$$S + T = \frac{1}{1 + D_{cl}GH} + \frac{FD_{cl}G}{1 + D_{cl}GH}$$

$$= \frac{1 + FD_{cl}G}{1 + D_{cl}GH}$$

$$= 1 \quad \text{if } F = H. \tag{W4.5}$$

(b) Applying the formula for sensitivity of T to changes in D_{cl}:

$$S_T^D = D_{cl}\frac{1 + D_{cl}GH}{FD_{cl}G}\frac{(1 + D_{cl}GH)FG - FD_{cl}G(GH)}{(1 + D_{cl}GH)^2}$$

$$= \frac{1}{1 + D_{cl}GH}.$$

This is not surprising as D_{cl} and G are in series.

(c) Applying the formula for sensitivity of T to changes in F:

$$S_T^F = F\frac{1 + D_{cl}GH}{FD_{cl}G}\frac{(1 + D_{cl}GH)(D_{cl}G)}{(1 + D_{cl}GH)^2}$$

$$\frac{1 + D_{cl}GH}{1 + D_{cl}GH}$$

$$= 1.$$

In this case, the F term is in the open loop, so it has a sensitivity of unity.

(d) Applying the formula for sensitivity of T to changes in H,

$$S_T^H = H\frac{1 + D_{cl}GH}{FD_{cl}G}\frac{(1 + D_{cl}GH)0 - FD_{cl}G(D_{cl}G)}{(1 + D_{cl}GH)^2}$$

$$= -\frac{D_{cl}GH}{(1 + D_{cl}GH)}.$$