Appendix WC
Controllability and Observability

Controllability and observability are important structural properties of dynamic systems. First identified and studied by Kalman (1960) and later by Kalman et al. (1961), these properties have continued to be examined during the last five decades. We will discuss only a few of the known results for linear constant systems with one input and one output. In the text, we discuss these concepts in connection with control law and estimator designs. For example, in Section 7.4, we suggest that, if the square matrix given by

$$\mathcal{C} = [\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \ldots\ \mathbf{A}^{n-1}\mathbf{B}], \tag{WC.1}$$

is nonsingular, by transformation of the state we can convert the given description into control-canonical form. We can then construct a control law that will give the closed-loop system an arbitrary characteristic equation.

WC.1 Controllability

We begin our formal discussion of controllability with the first of four definitions.

Definition WC.1 *The system* $(\mathbf{A}, \mathbf{B})$ *is* ***controllable*** *if, for any given nth-order polynomial* $\alpha_c(s)$*, there exists a (unique) control law* $u = -\mathbf{Kx}$ *such that the characteristic polynomial of* $\mathbf{A} - \mathbf{BK}$ *is* $\alpha_c(s)$.

From the results of Ackermann's formula (see Appendix WD), we have the following mathematical test for controllability: $(\mathbf{A}, \mathbf{B})$ is a controllable pair if and only if the rank of $\mathcal{C}$ is n. Definition WC.1 based on pole placement is a frequency-domain concept. Controllability can be equivalently defined in the time domain.

Definition WC.2 *The system* $(\mathbf{A}, \mathbf{B})$ *is* ***controllable*** *if there exists a (piecewise continuous) control signal* $u(t)$ *that will take the state of the system from any initial state* $\mathbf{x}_0$ *to any desired final state* $\mathbf{x}_f$ *in a finite time interval.*

We will now show that the system is controllable by this definition if and only if $\mathcal{C}$ is full rank. We first assume that the system is controllable but

$$\text{rank}[\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \ldots\ \mathbf{A}^{n-1}\mathbf{B}] < n. \tag{WC.2}$$

We can then find a vector $\mathbf{v}$ such that

$$\mathbf{v}[\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \dots\ \mathbf{A}^{n-1}\mathbf{B}] = 0 \tag{WC.3}$$

or

$$\mathbf{vB} = \mathbf{vAB} = \mathbf{vA}^2\mathbf{B} = \dots = \mathbf{vA}^{n-1}\mathbf{B} = 0. \tag{WC.4}$$

The Cayley-Hamilton theorem states that $\mathbf{A}$ satisfies its own characteristic equation, namely,

$$-\mathbf{A}^n = a_1\mathbf{A}^{n-1} + a_2\mathbf{A}^{n-2} + \dots + a_n\mathbf{I}. \tag{WC.5}$$

Therefore,

$$-\mathbf{vA}^n\mathbf{B} = a_1\mathbf{vA}^{n-1}\mathbf{B} + a_2\mathbf{vA}^{n-2}\mathbf{B} + \dots + a_n\mathbf{vB} = 0. \tag{WC.6}$$

By induction, $\mathbf{vA}^{n+k}\mathbf{B} = 0$ for $k = 0, 1, 2, \dots$, or $\mathbf{vA}^m\mathbf{B} = 0$ for $m = 0, 1, 2, \dots$, and thus

$$\mathbf{v}e^{\mathbf{A}t}\mathbf{B} = \mathbf{v}\left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \dots\right)\mathbf{B} = 0, \tag{WC.7}$$

for all t. However, the zero initial-condition response ($\mathbf{x}_0 = \mathbf{0}$) is

$$\begin{aligned}\mathbf{x}(t) &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\,d\tau \\ &= e^{\mathbf{A}t}\int_0^t e^{-A\tau}\mathbf{B}u(\tau)\,d\tau.\end{aligned} \tag{WC.8}$$

Using Eq. (WC.7), Eq. (WC.8) becomes

$$\mathbf{vx}(t) = \int_0^t \mathbf{v}e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\,d\tau = 0, \tag{WC.9}$$

for all $u(t)$ and $t > 0$. This implies that all points reachable from the origin are orthogonal to $\mathbf{v}$. This restricts the reachable space, and therefore contradicts the second definition of controllability. Thus if $\mathcal{C}$ is singular, $(\mathbf{A}, \mathbf{B})$ is not controllable by Definition WC.2.

Next, we assume $\mathcal{C}$ is full rank but $(\mathbf{A}, \mathbf{B})$ is uncontrollable by Definition WC.2. This means that there exists a nonzero vector $\mathbf{v}$ such that

$$\mathbf{v}\int_0^{t_f} e^{\mathbf{A}^{(t_f-\tau)}}\mathbf{B}u(\tau)\,d\tau = 0, \tag{WC.10}$$

because the whole state-space is not reachable. But Eq. (WC.10) implies that

$$\mathbf{v}e^{\mathbf{A}^{(t_f-\tau)}}\mathbf{B} = 0, \quad 0 \le \tau \le t_f. \tag{WC.11}$$

If we set $\tau = t_f$, we see $\mathbf{vB} = 0$. Also, differentiating Eq. (WC.11) and letting $\tau = t_f$ gives $\mathbf{vAB} = 0$. Continuing this process, we find

$$\mathbf{vB} = \mathbf{vAB} = \mathbf{vA}^2\mathbf{B} = \dots = \mathbf{vA}^{n-1}\mathbf{B} = 0, \tag{WC.12}$$

which contradicts the assumption that $\mathcal{C}$ is full rank.

We have now shown the system is controllable by Definition WC.2 if and only if the rank of $\mathcal{C}$ is n, which is exactly the same condition we found for pole assignment.

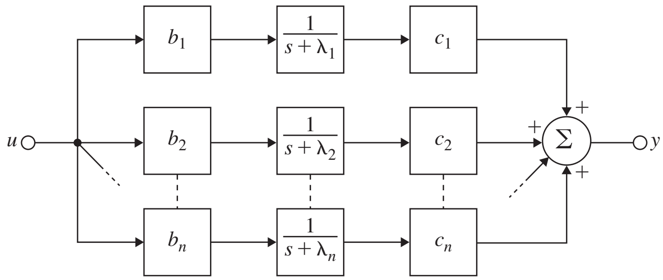

Figure WC.1
Block diagram of a system with a diagonal matrix

Our final definition comes closest to the structural character of controllability.

Definition WC.3 *The system* $(\mathbf{A}, \mathbf{B})$ *is* ***controllable*** *if every mode of* $\mathbf{A}$ *is connected to the control input.*

Because of the generality of the modal structure of systems, we will treat only the case of systems for which $\mathbf{A}$ can be transformed to diagonal form. (The double-integration plant does *not* qualify.) Suppose we have a diagonal matrix $\mathbf{A}_d$ and its corresponding input matrix $\mathbf{B}_d$ with elements b_i. The structure of such a system is shown in Fig. (WC.1). By definition, for a controllable system, the input must be connected to each mode so the b_i are all nonzero. However, this is not enough if the poles (λ_i) are not distinct. Suppose, for instance, that $\lambda_1 = \lambda_2$. The first two state equations are then

$$\begin{aligned} \dot{x}_{1d} &= \lambda_1 x_{1d} + b_1 u, \\ \dot{x}_{2d} &= \lambda_1 x_{2d} + b_2 u. \end{aligned} \tag{WC.13}$$

If we define a new state, $\xi = b_2 x_{1d} - b_1 x_{2d}$, the equation for ξ is

$$\dot{\xi} = b_2 \dot{x}_{1d} - b_1 \dot{x}_{2d} = b_2 \lambda_1 x_{1d} + b_2 b_1 u - b_1 \lambda_1 x_{2d} - b_1 b_2 u = \lambda_1 \xi, \tag{WC.14}$$

which does not include the control u; hence, ξ is not controllable. The point is that if any two poles are equal *in a diagonal* $\mathbf{A}_d$ *system with only one input*, we effectively have a hidden mode that is not connected to the control, and the system is not controllable (see Fig. WC.2a). This is because the two state variables move together exactly, so we cannot

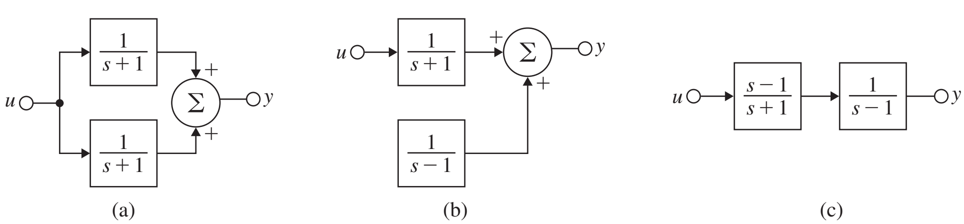

Figure WC.2
Examples of uncontrollable systems

independently control x_{1d} and x_{2d}. Therefore, even in such a simple case, we have two conditions for controllability:

1. All eigenvalues of $\mathbf{A}_d$ are distinct.
2. No element of $\mathbf{B}_d$ is zero.

Now let us consider the controllability matrix of this diagonal system. By direct computation,

$$\mathcal{C} = \begin{bmatrix} b_1 & b_1\lambda_1 & \dots & b_1\lambda_1^{n-1} \\ b_2 & b_2\lambda_2 & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ b_n & b_n\lambda_n & \dots & b_n\lambda_n^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} b_1 & & & \mathbf{0} \\ & b_2 & & \\ & & \ddots & \\ & \mathbf{0} & & b_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix}. \tag{WC.15}$$

Note that the controllability matrix $\mathcal{C}$ is the product of two matrices and is nonsingular if and only if both of these matrices are invertible. The first matrix has a determinant that is the product of b_i, and the second matrix (called a Vandermonde matrix) is nonsingular if and only if the λ_i are distinct. Thus, Definition WC.3 is equivalent to having a nonsingular $\mathcal{C}$ also.

Important to the subject of controllability is the **Popov–Hautus–Rosenbrock (PHR) test** (see Rosenbrock, 1970, and Kailath, 1980), which is an alternate way to test the rank (or determinant) of $\mathcal{C}$. The system $(\mathbf{A}, \mathbf{B})$ is controllable if the system of equations

$$\mathbf{v}^T[s\mathbf{I} - \mathbf{A} \quad \mathbf{B}] = \mathbf{0}^T, \tag{WC.16}$$

has only the trivial solution $\mathbf{v}^T = \mathbf{0}^T$—that is, if the **matrix pencil**

$$\text{rank}\,[s\mathbf{I} - \mathbf{A} \quad \mathbf{B}] = n, \tag{WC.17}$$

is full rank for all s, or if there is no nonzero $\mathbf{v}^T$ such that[1]

$$\mathbf{v}^T\mathbf{A} = s\mathbf{v}^T, \tag{WC.18}$$

$$\mathbf{v}^T\mathbf{B} = 0. \tag{WC.19}$$

This test is equivalent to the rank-of-$\mathcal{C}$ test. It is easy to show that, if such a vector $\mathbf{v}$ exists, $\mathcal{C}$ is singular. For if a nonzero $\mathbf{v}$ exists such that $\mathbf{v}^T\mathbf{B} = 0$, then by Eqs. (WC.18) and (WC.19), we have

$$\mathbf{v}^T\mathbf{AB} = s\mathbf{v}^T\mathbf{B} = 0. \tag{WC.20}$$

Then, multiplying by $\mathbf{AB}$, we find that

$$\mathbf{v}^T\mathbf{A}^2\mathbf{B} = s\mathbf{v}^T\mathbf{AB} = 0, \tag{WC.21}$$

[1] $\mathbf{v}^T$ is a left eigenvector of $\mathbf{A}$.

and so on. Thus we determine that $\mathbf{v}^T\mathcal{C} = \mathbf{0}^T$ has a nontrivial solution, that $\mathcal{C}$ is singular, and that the system is not controllable. To show that a nontrivial $\mathbf{v}^T$ exists if $\mathcal{C}$ is singular requires more development, which we will not give here (see Kailath, 1980).

We have given two pictures of uncontrollability. Either a mode is physically disconnected from the input (see Fig. WC.2b), or else two parallel subsystems have identical characteristic roots (see Fig. WC.2a). The control engineer should be aware of the existence of a third simple situation, as illustrated in Fig. WC.2c, namely, a **pole-zero cancellation**. Here the problem is the mode at $s = 1$ appears to be connected to the input, but is masked by the zero at $s = 1$ in the preceding subsystem; the result is an uncontrollable system. This can be confirmed in several ways. First let us look at the controllability matrix. The system matrices are

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

so the controllability matrix is

$$\mathcal{C} = [\ \mathbf{B} \quad \mathbf{AB}\ ] = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \tag{WC.22}$$

which is clearly singular. The controllability matrix may be computed using the `ctrb` command in Matlab: `[cc]=ctrb(A,B)`. If we compute the transfer function from u to x_2, we find

$$\mathbf{H}(s) = \frac{s-1}{s+1}\left(\frac{1}{s-1}\right) = \frac{1}{s+1}. \tag{WC.23}$$

Because the natural mode at $s = 1$ disappears from the input–output description, it is not connected to the input. Finally, if we consider the **PHR** test,

$$[s\mathbf{I} - \mathbf{A} \quad \mathbf{B}] = \begin{bmatrix} s+1 & 0 & -2 \\ -1 & s-1 & 1 \end{bmatrix}, \tag{WC.24}$$

and let $s = 1$, then we must test the rank of

$$\begin{bmatrix} 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix},$$

which is clearly less than 2. This result means, again, that the system is uncontrollable.

Definition WC.4 *The asymptotically stable system (*$\mathbf{A}, \mathbf{B}$*) is* ***controllable*** *if the controllability Gramian, the square symmetric matrix* $\mathcal{C}_g$*, given by the solution to the Lyapunov equation*

$$\mathbf{A}\mathcal{C}_g + \mathcal{C}_g\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{0}, \tag{WC.25}$$

is nonsingular. The controllability ***Gramian*** *is also the solution to the following integral equation:*

$$\mathcal{C}_g = \int_0^\infty e^{\tau\mathbf{A}}\mathbf{B}\mathbf{B}^T e^{\tau\mathbf{A}^T}\,d\tau. \tag{WC.26}$$

One physical interpretation of the controllability Gramian is that, if the input to the system is white Gaussian noise, $\mathcal{C}_g$ is the covariance of the state. The controllability Gramian (for an asymptotically stable system) can be computed with the following command in Matlab: $[\mathsf{cg}] = \mathsf{gram}(\mathsf{A}, \mathsf{B})$.

In conclusion, the four definitions for controllability—pole assignment (Definition WC.1), state reachability (Definition WC.2), mode coupling to the input (Definition WC.3), and controllability Gramian (Definition WC.4)—are equivalent. The tests for any of these four properties are found in terms of the rank of the controllability, controllability Gramian matrices, or the rank of the **matrix pencil** $[s\mathbf{I} - \mathbf{A}\ \mathbf{B}]$. If $\mathcal{C}$ is nonsingular, we can assign the closed-loop poles arbitrarily by state feedback, we can move the state to any point in the state space in a finite time, and every mode is connected to the control input.[2]

WC.2 Observability

So far, we have discussed only controllability. The concept of observability is parallel to that of controllability, and all of the results we have discussed thus far may be transformed to statements about observability by invoking the property of duality, as discussed in Section 7.7.2. The observability definitions are analogous to those for controllability.

Definition WC.1: The system ($\mathbf{A}$, $\mathbf{C}$) is **observable** if, for any nth-order polynomial $\alpha_e(s)$, there exists an estimator gain $\mathbf{L}$ such that the characteristic equation of the state estimator error is $\alpha_e(s)$.
Definition WC.2: The system ($\mathbf{A}$, $\mathbf{C}$) is **observable** if, for any $\mathbf{x(0)}$, there is a finite time τ such that $\mathbf{x(0)}$ can be determined (uniquely) from $u(t)$ and $y(t)$ for $0 \le t \le \tau$.
Definition WC.3: The system ($\mathbf{A}$, $\mathbf{C}$) is **observable** if every dynamic mode in $\mathbf{A}$ is connected to the output through $\mathbf{C}$.
Definition WC.4: The asymptotically stable system ($\mathbf{A}$, $\mathbf{C}$) is **observable** if the observability Gramian is nonsingular.

As we saw in the discussion for controllability, mathematical tests can be developed for observability. The system is observable if the observability matrix

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}, \tag{WC.27}$$

is nonsingular. If we take the transpose of $\mathcal{O}$, and let $\mathbf{C}^T = \mathbf{B}$ and $\mathbf{A}^T = \mathbf{A}$, then we find the controllability matrix of ($\mathbf{A}$, $\mathbf{B}$), which is another manifestation of duality. The observability matrix $\mathcal{O}$ may be

[2] We have shown the latter for diagonal $\mathbf{A}$ only, but the result is true in general.

computed using the `obsv` command in Matlab: `[oo]=obsv(A,C)`. The system $(\mathbf{A}, \mathbf{C})$ is observable if the following **matrix pencil** is full rank for all s:

$$\operatorname{rank}\begin{bmatrix} s\mathbf{I}-\mathbf{A} \\ \mathbf{C} \end{bmatrix} = n. \tag{WC.28}$$

The observability Gramian $\mathcal{O}_g$, which is a symmetric matrix, and the solution to the integral equation

$$\mathcal{O}_g = \int_0^\infty e^{\tau\mathbf{A}^T}\mathbf{C}^T\mathbf{C}e^{\tau\mathbf{A}}\,d\tau, \tag{WC.29}$$

as well as the Lyapunov equation

$$\mathbf{A}^T\mathcal{O}_g + \mathcal{O}_g\mathbf{A} + \mathbf{C}^T\mathbf{C} = \mathbf{0}, \tag{WC.30}$$

also can be computed (for an asymptotically stable system) using the `gram` command in Matlab: `[og]=gram(A',C')`. The observability Gramian has an interpretation as the "information matrix" in the context of estimation.