Appendix W4.2.2.1
Truxal's Formula for the Error Constants

Truxal (1955) derived a formula for the velocity constant of a Type 1 system in terms of the closed-loop poles and zeros, which is a formula that connects the steady-state error to the system's dynamic response. Since control design often requires a trade-off between these two characteristics, Truxal's formula can be useful to know. Its derivation is quite direct. Suppose the closed-loop transfer function $\mathcal{T}(s)$ of a Type 1 system is

$$\mathcal{T}(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}. \tag{W4.6}$$

Since the steady-state error in response to a step input in a Type 1 system is zero, the DC gain is unity; thus,

$$\mathcal{T}(0) = 1. \tag{W4.7}$$

The system error is given by

$$E(s) \triangleq R(s) - Y(s) = R(s)\left[1 - \frac{Y(s)}{R(s)}\right] = R(s)[1-\mathcal{T}(s)]. \tag{W4.8}$$

The system error due to a unit ramp input is given by

$$E(s) = \frac{1-\mathcal{T}(s)}{s^2}. \tag{W4.9}$$

Using the Final Value Theorem, we get

$$e_{ss} = \lim_{s\to 0}\frac{1-\mathcal{T}(s)}{s}. \tag{W4.10}$$

Using L'Hôpital's rule, we rewrite Eq. (W4.10) as

$$e_{ss} = -\lim_{s\to 0}\frac{d\mathcal{T}}{ds}, \tag{W4.11}$$

or

$$e_{ss} = -\lim_{s\to 0}\frac{d\mathcal{T}}{ds} = \frac{1}{K_v}. \tag{W4.12}$$

Equation (W4.12) implies that $1/K_v$ is related to the slope of the transfer function at the origin, which is a result shown in Section 6.1.2. Using Eq. (W4.7), we can rewrite Eq. (W4.12) as

$$e_{ss} = -\lim_{s\to 0}\frac{d\mathcal{T}}{ds}\frac{1}{\mathcal{T}}, \tag{W4.13}$$

or

$$e_{ss} = -\lim_{s\to 0}\frac{d}{ds}[\ln\mathcal{T}(s)]. \tag{W4.14}$$

Substituting Eq. (W4.6) into Eq. (W4.14), we get

$$e_{ss} = -\lim_{s\to 0} \frac{d}{ds}\left\{\ln\left[K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}\right]\right\}, \tag{W4.15}$$

$$= -\lim_{s\to 0} \frac{d}{ds}\left[\ln K + \sum_{i=1}^{m}\ln(s-z_i) - \sum_{i=1}^{m}\ln(s-p_i)\right], \tag{W4.16}$$

or

$$\frac{1}{K_v} = -\frac{d\ln\mathcal{T}}{ds}\bigg|_{s=0} = \sum_{i=1}^{n} -\frac{1}{p_i} + \sum_{i=1}^{m}\frac{1}{z_i}. \tag{W4.17}$$

We observe from Eq. (W4.17) that K_v increases as the closed-loop poles move away from the origin. Similar relationships exist for other error coefficients, and these are explored in the problems.

EXAMPLE W4.2 *Truxal's Formula*

Truxal's formula

A third-order Type 1 system has closed-loop poles at $-2 \pm 2j$ and -0.1. The system has only one closed-loop zero. Where should the zero be if a $K_v = 10\ \text{sec}^{-1}$ is desired?

Solution. From Truxal's formula, we have

$$\frac{1}{K_v} = -\frac{1}{-2+2j} - \frac{1}{-2-2j} - \frac{1}{-0.1} + \frac{1}{z},$$

or

$$0.1 = 0.5 + 10 + \frac{1}{z},$$

$$\frac{1}{z} = 0.1 - 0.5 - 10$$

$$= -10.4.$$

Therefore, the closed-loop zero should be at $z = 1/-10.4 = -0.096$.