Tutorial on Robust Tracking and Regulation

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Abstract

Two robust methods are described for designing a control system to track a persistent reference, such as a constant or a sine wave, while rejecting a disturbance of the same type. In the first case, the Internal Model principle (IM) is developed and the design is essentially an extended control with the addition of the IM. In the second case, the dual view is taken and an Extended Estimator (Xest) is designed that will do the same thing. A third method for tracking, which is open loop and not robust, is known as Model Following. This method is also developed and the results compared to the other methods. This note is intended as a tutorial giving simple derivations of the methods and a comparison of both their structure and their performance.

1 Introduction

Two robust methods to track a persistent reference and eliminate a persistent disturbance are the Internal Model method and the Extended Estimator method. The use of integral action to track a constant reference or to eliminate a constant bias has been part of the tools of control from the earliest times. This concept was generalized as the Internal Model principle by Wonham and Francis in 1975.[1] In another development, the concept of an estimator of the state of a dynamic plant having random inputs was introduced by Kalman and Bucy in 1961.[2] The deterministic case was introduced by Luenberger in 1964.[3] One of the basic assumptions of the Kalman Filter is that the disturbance signal is a white noise, having a constant spectrum. If the actual disturbance signal has a shaped spectrum, it is said to be 'colored' and is modeled as a white signal shaped by a dynamic system or filter. Construction of the Kalman filter then

[1] The Internal Model Principle of Control Theory, B.A. Francis and W. M. Wonham Automatica, Vol. 12, 1975.

[2] New Results in Linear Filtering and Prediction Theory, R, E. Kalman and R. S. Bucy, Journal of Basic Engineering, March 1961

[3] Observing the State of a Linear System, D. G Luenberger, IEEE Transactions on Military Electronics, vol. 8, 1964

requires that the state of the shaping filter is estimated along with the state of the plant. The Extended Estimator is the deterministic version of this case. While these methods are widely used, a simple derivation of their structure and a comparison of their properties is not readily available. In the Section 2, the equations and block diagram of the internal model are given. In the Section 3 the same is done for the extended estimator. In the Section 4 the frequency response is used to illustrate the designs in a classical way. Worked out examples of both methods are presented in Section 5, sample problems are given in Section 6 and the Appendix briefly describes the related, but none robust method of Model Following. Programs in MATLAB and SIMULINK used to make the calculations are available on line at the site fpe6e.com.

2 Design of the Internal Model.

The idea of an internal model is quite simple: if one wants the plant output to track a sinusoid with *no* error in the steady state, then the plant must be driven by a sinusoidal signal that will cause the plant output to produce the sine wave with *zero* input. Such a signal can be generated by an oscillator of the required frequency, which is said to be an *internal model* of the reference generator. Notice that for this method to work, the plant transfer function must not block the oscillator signal by having a zero at the relevant frequency. By equations one can see the same thing. Suppose a plant with transfer function $\frac{b(s)}{a(s)}$ has a serial controller $\frac{c(s)}{d(s)}$ and is asked to track *exactly* a reference input described by $\frac{p(s)}{q(s)}$. With unity feedback, the transform of the error for this system will be

$$E(s) = \frac{p(s)}{q(s)} \frac{a(s)d(s)}{a(s)d(s) + b(s)c(s)} \tag{1}$$

If this transform is to have no residue at a root of $q(s)$ then that polynomial must be canceled from the error transform. If it is canceled by the controller denominator, $d(s)$, then the controller is said to include an internal model. If it is canceled by the plant transfer function denominator $a(s)$ then the plant is said to include an *implicit* internal model. This is most often the case when the plant transfer function includes integral action, an implicit model for steps. The tracking result holds even with changes in the plant parameters just so long as the system remains stable. Thus the solution is said to be *robust.* However, with an *implicit* internal model, the poles of the plant used to cancel the input poles *cannot* change if the tracking is to be accurate Notice that if not only the internal model denominator, $d(s)$ but also the plant numerator, $b(s)$ has the specified zeros, then they will cancel in the overall transfer function and will *not* be available to cancel the $q(s)$. We note here that if the system is not unity feedback or has a disturbance which we wish to eliminate exactly, then the error transform will differ from the above so that although the concept is the same, different conclusions will need to be made.

The Internal Model system can be designed in a set of simple steps. We begin with the nth order system equations for the plant:

$$\begin{aligned} \dot{\mathbf{x}} &= F\mathbf{x} + Gu + G_w w && (2) \\ e &= r - y && (3) \\ &= r - H\mathbf{x} && (4) \end{aligned}$$

For this problem, using $'p'$ as the differential operator, we assume that there is an mth degree scalar polynomial, $d(p)$ such that $d(p)r(t) = 0$. and $d(p)w(t) = 0$ For example, if $w = 1$ and $r = \sin(t)$, then $d(p) = p^3 + p$. If we operate on Eq.(2) and Eq. (3) with $d(p)$ the reference and the disturbance are eliminated and the result is

$$\begin{aligned} d(p)\dot{\mathbf{x}} &= Fd(p)\mathbf{x} + Gd(p)u && (5) \\ d(p)e &= -d(p)y = -Hd(p)\mathbf{x} \end{aligned}$$

If we now define new variables as $d(p)\mathbf{x} = \mathbf{z}$ and $d(p)u = \mu$, then the equations become:

$$\begin{aligned} \dot{\mathbf{z}} &= F\mathbf{z} + G\mu && (6) \\ e &= \frac{-H\mathbf{z}}{d(p)} \end{aligned}$$

Eq.(6) represents two systems in series, as shown in Fig. 1. The error is the output of a system that can be described in control canonical form by the state parameters A, B, and has the transfer function $\frac{-1}{d(s)}$. The state of this system is comprised of the error and its derivatives as $\boldsymbol{\eta} = [\ e^{(m)} \quad e^{(m-1)} \quad \ldots \quad \dot{e} \quad e\ ]^T$

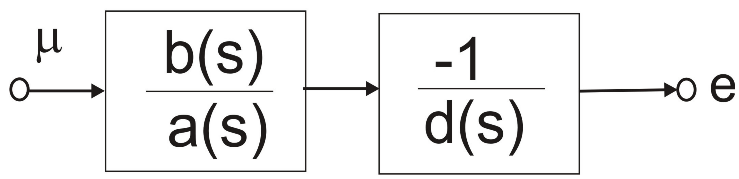

Figure 1: Block diagram for Internal Model design

To stabilize this overall system with state feedback, we select $n + m$ control poles in a vector p_{ci}, and compute control gains $[\ K_{zi} \quad K_{\eta i}\ ]$ using the MATLAB function *place* on the composite system which is described by the equations:

$$\begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} F & 0 \\ BH & A \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \boldsymbol{\eta} \end{bmatrix} + \begin{bmatrix} G \\ 0 \end{bmatrix} \mu$$

with the result that $\mu = -\begin{bmatrix} K_{zi} & K_{\eta i} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \boldsymbol{\eta} \end{bmatrix}$ as shown in Fig.2.

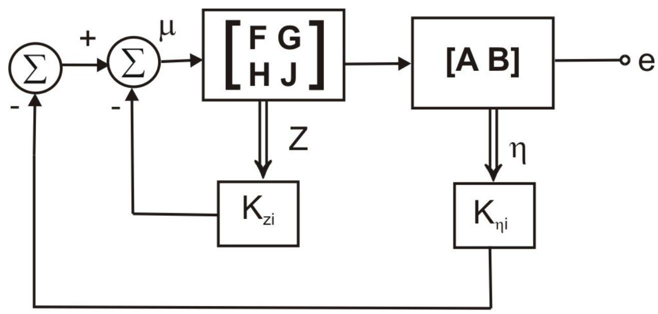

Figure 2: Block diagram showing the state feedback for the composite system

In view of the nature of the state $\boldsymbol{\eta}$, as composed of the error and its derivatives, a polynomial in p can be defined, based on $K_{\eta i}$, as $c(p)$ such that $\mu = -K_{zi}\mathbf{z} + c(p)e$. We can now unscramble the equations to recover the original variables. Starting with Eq.(6), dividing by $d(p)$, and using the just derived control law, we get

$$\begin{aligned}\dot{\mathbf{x}} &= F\mathbf{x} + Gu + G_w w \\ u &= -K_{zi}\mathbf{x} + \frac{c(p)}{d(p)}e \\ &= -K_{zi}\mathbf{x} + \frac{c(p)}{d(p)}(r - H\mathbf{x})\end{aligned}$$

Notice that in these equations we have restored both reference and disturbance because dividing by $d(p)$ we get, for example, $\frac{d(p)}{d(p)}w$ which must be taken as w. The result, with the Internal Model shown, is sketched in Fig.3.

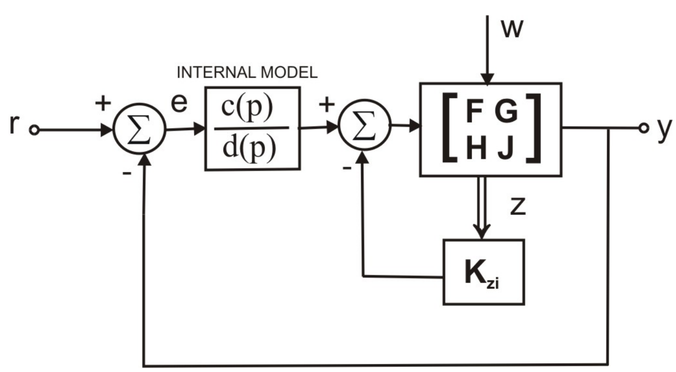

Figure 3: Block diagram of the Internal Model design

Finally, since the plant state is rarely available, we replace the plant state feedback with feedback of the estimated state, as shown in Fig. 4. The standard state estimator equations are

$$\begin{aligned} \frac{d\hat{\mathbf{x}}}{dt} &= F\hat{\mathbf{x}} + Gu + L_{\mathbf{xi}}(y - \hat{y}) \\ \hat{u} &= -K_{zi}\hat{\mathbf{x}} + \frac{c(p)}{d(p)}e \end{aligned}$$

Notice that we have included saturation of the control signal in this structure. Before we give an example of this design, it is important to notice that the control law shown in Fig.2 guarantees that the *modified* state and controls as well as the error go to zero in the steady state but that these are *not* the physical state, $\mathbf{x}$ or the physical control, u. It is $\mathbf{z} = d(p)\mathbf{x}$ and $\mu = d(p)u$ in addition the to physical error that are guaranteed to be sent to zero.

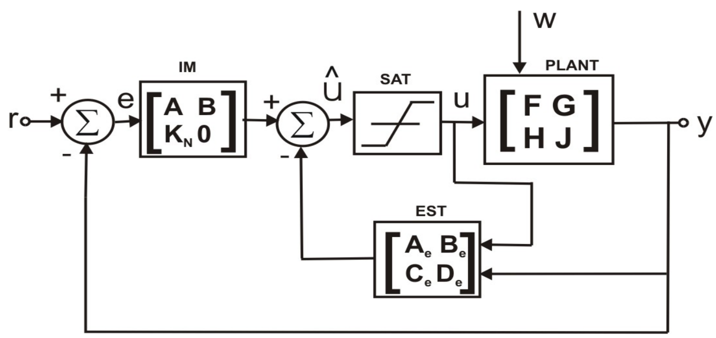

Figure 4: Block Diagram of the final Internal Model system

3 The Extended Estimator

As a second approach to the design of robust control with external inputs, we develop a method for tracking a reference input and rejecting disturbances by an Extended Estimator rather than with an explicit Internal Model. The internal model will show up as part of the estimator in this case. The method is based on augmenting the usual state estimator to include estimates of the external signals in a way that permits us to cancel out their effects on the system error asymptotically. The physical situation is sketched in Figure 5 showing the plant with disturbance, w, introduced into the plant and a reference at the output.

Taking the difference between reference and plant output, the system output is the error, e. In order to cancel the effects of the reference and the disturbance as well, an equivalent input is introduced as shown in Figure 6. This equivalent external signal generator system is described by the matrices A, B, C, D. That is to say, the dynamics of the system matrix A, with suitable initial conditions, can reproduce the effects of both the disturbance and the reference input so as to produce the actual error signal at the output of the equivalent plant. For

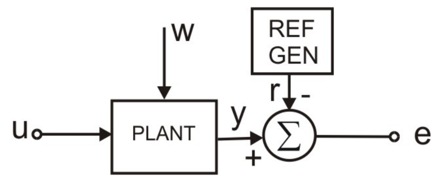

Figure 5: Original system for design of Extended Estimator

this to work, as was the case with the internal model design, the plant must not have a zero at any of the eigenvalues of A as that would prevent that portion of the equivalent input from getting to the output. For notation, we define the state of the equivalent input generator as $\boldsymbol{\eta}$ and its output as $\rho = C\boldsymbol{\eta}$. Notice in particular that the state of the 'Plant' in this set-up is *not* the state of the physical plant even though it is described by the same dynamics. Furthermore, it has a non-physical input although its output is the true system *error*, which is *not* the usual plant output, which is y.

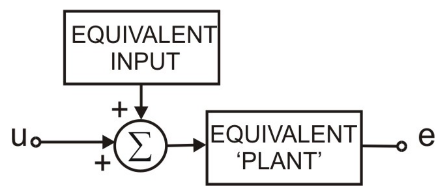

Figure 6: Equivalent system for desgn of the Extended Estimator

This equivalent system is clearly not controllable, as the control signal has no influence on the equivalent external input signal. The design plan is to first design a simple feedback control for the 'plant' part of the system alone and then to build an *extended estimator* that will provide an estimate of the complete system state, including both the 'plant' state and the state of the external input generator system. This latter estimate will be used to construct the control signal which will make the system stable as well as *cancel* the equivalent external input asymptotically. The first step is to design the plant control law based on the equivalent 'plant' state, $\mathbf{z}$. For this we select a suitable set of control poles as p_{cx} and compute $K_{zx} = place(F, G, p_{cx})$ and apply it to the plant as shown in Figure 7.

Next we estimate the combined state of this observable, composite, equivalent system using the available system error signal. We will use these extended estimates to compute the control signal. We must emphasize here that the estimate of the state in this equivalent world is *not* an estimate of the physical plant state but is of a state that produces the same *error* as the physical system when the effects of both reference and disturbance external inputs are generated at the 'plant' input. We emphasize that the estimates are constructed from the system *error*, e, not from the physical plant output, y. The equations on which

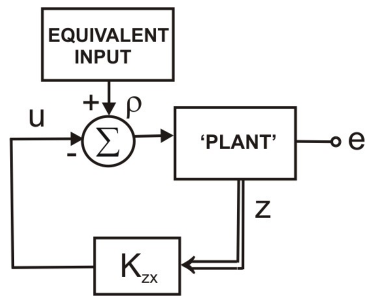

Figure 7: Control law block diagram for the Extended Estimator

the design is based are thus

$$\begin{aligned}\dot{\mathbf{z}} &= F\mathbf{z} + Gu + GC\boldsymbol{\eta} \\ \dot{\boldsymbol{\eta}} &= A\boldsymbol{\eta} \\ e &= H\mathbf{z}\end{aligned} \tag{7}$$

An estimator for this system is described by the standard equations:

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{z}} &= F\hat{\mathbf{z}} + Gu + GC\hat{\boldsymbol{\eta}} + \mathbf{L}_{zx}(e - H\hat{\mathbf{z}}) \\ \frac{d}{dt}\hat{\boldsymbol{\eta}} &= A\hat{\boldsymbol{\eta}} + \mathbf{L}_{nx}(e - H\hat{\mathbf{z}})\end{aligned} \tag{8}$$

and the control is computed by the formula

$$u = -K_{zx}\hat{\mathbf{z}} - C\hat{\boldsymbol{\eta}}$$

The control equation includes the usual control law, $-K_{zx}\hat{\mathbf{z}}$ as well as the estimate of the output of the equivalent input generator, $C\hat{\boldsymbol{\eta}}$. This latter term will be used to cancel the effects of the external signals in the steady state, as we will see presently. The estimator law, $Lt = \begin{bmatrix} L_{zx} & L_{nx} \end{bmatrix}'$, is designed by selecting $n + m$ estimator poles, p_{ex}.and using *place* in the usual way. From these results we construct the system error equations as

$$\begin{aligned}\frac{d\tilde{\mathbf{z}}}{dt} &= F\tilde{\mathbf{z}} + GC\tilde{\boldsymbol{\eta}} - L_{zx}H\tilde{\mathbf{z}} \\ \frac{d\tilde{\boldsymbol{\eta}}}{dt} &= A\tilde{\boldsymbol{\eta}} - L_{nx}H\tilde{\mathbf{z}}\end{aligned}$$

Based on the design of the estimator gain Lt, these equations are stable and the entire estimator error state will go to zero asymptotically. It is informative to rewrite the estimator equations as the system controller, having input e and output u.

$$\frac{d}{dt}\begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} F - L_{zx}H - GK_{zx} & 0 \\ L_{nx}H & A \end{bmatrix}\begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\boldsymbol{\eta}} \end{bmatrix} + \begin{bmatrix} L_{zx} \\ L_{nx} \end{bmatrix} e$$
$$u = -K_{zx}\hat{\mathbf{z}} - C\hat{\boldsymbol{\eta}}$$

From these equations, it is clear that the eigenvalues of the controller include those of the matrix A, which is to say the controller contains an internal model of all the external inputs. The equations for the equivalent plant are given by, in turn,

$$\begin{aligned}
\dot{\mathbf{z}} &= F\mathbf{z} + Gu + GC\boldsymbol{\eta} \\
\dot{\mathbf{z}} &= F\mathbf{z} - G(K_{zx}\hat{\mathbf{z}} + C\hat{\boldsymbol{\eta}}) + GC\boldsymbol{\eta} \\
\dot{\mathbf{z}} &= F\mathbf{z} - GK_{zx}(\mathbf{z} - \tilde{\mathbf{z}}) - GC\hat{\boldsymbol{\eta}} + GC\boldsymbol{\eta} \\
\dot{\mathbf{z}} &= (F - GK_{zx})\mathbf{z} + GK_{zx}\tilde{\mathbf{z}} + GC\tilde{\boldsymbol{\eta}}
\end{aligned}$$

In view of the fact that K_{zx} was designed to make $F - GK_{zx}$ stable and that we have shown that both $\tilde{\mathbf{z}}$ and $\tilde{\boldsymbol{\eta}}$ go to zero, we can conclude that $\mathbf{z}$ and thus $e = H\mathbf{z}$ will go to zero, which is the object of all this, after all! Notice that this result is robust with respect to the plant parameters, F, G, H, but not with respect to the characteristic polynomial of A, which is the model polynomial, $d(p)$. Again, it is $\mathbf{z} = d(p)\mathbf{x}$ that goes to zero, not the physical state $\mathbf{x}$.The block diagram of the resulting *physical* system is shown in Fig 8 where we have included a saturation element at the (physical!) plant input. It is interesting to notice that by comparing Fig. 8 with Fig. 4 it can be seen that the Internal Model design results in a combined feedback-feed forward structure while the Extended Estimator results in a strictly feed forward structure. Thus, while they will both be robust in tracking, the transient responses will be different, often spectacularly so.

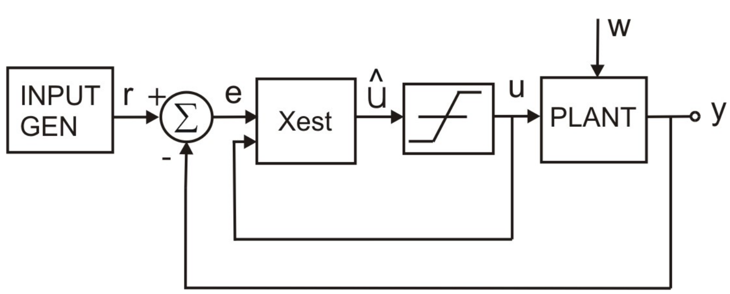

Figure 8: Final Block Diagram of the Extended Estimator Design

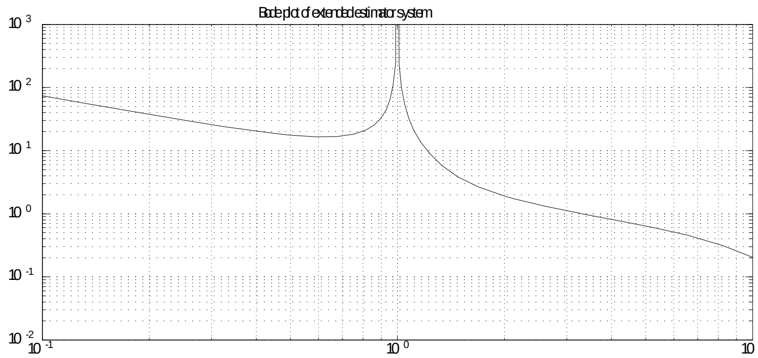

Figure 9: Fequency response of the extended estimator design for a sinusoidal disturbanace

4 Frequency Response

For the single-input-single-ouput case especially it is informative to consider the frequency response. For this purpose only the extended estimator is considered here but the internal model would show comparable results. As can be seen from Fig. 4, the structure of the extended estimator design is a unity-feedback system with plant P and controller C. As the external inputs are represented as signals at the control input, it is the loop gain from equivalent disturbance ρ to control u that is of most interest. This would be

$$U = -\frac{PC}{1+PC} W_e$$

From this formula, it is clear that the control will nearly match the equivalent disturbance over those frequencies for which the magnitude $|PC|$ is large. And example is given below of a system designed to reject a sinusoid at $1 rad/\sec$. The frequency response of PC is given in Fig. 9. Notice the extremely high gain at $1 rad/\sec$ generated by the oscillator comprising the internal model. It is this high gain that forces the error due to the external signal to go to zero.

In the paper[4] an alternative method of disturbance rejection is described. In this paper, it is not assummed that a state model of the disturbance is available but that the designer has only a known bandwidth of the disturbance signal. In fact, a careful reading of the paper reveals that while the method is phrased as based on disturbance estimation, the design seems to reproduce classical design in a fundamental way. For example, the purpose of the control in this case can be said to have one component to force the output to track the reference input

[4] *Improving Disturbance-Rejection Performance Based on an Equivalent-Input-Disturbance Approach*, by J. She, M. Fang, Y. Ohyama, H.Hashimoto, and M. Wu IEEE Trans. on Industrial Electronics, Vol. 55, No.1, 2008

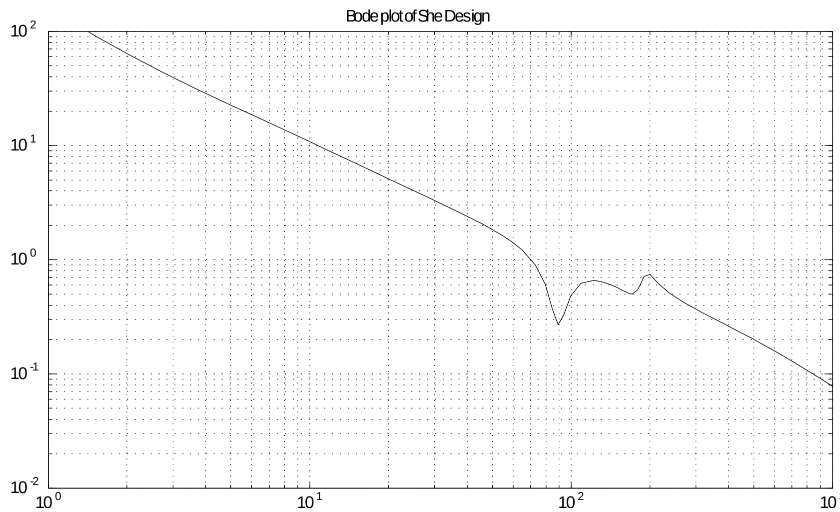

Figure 10: Frequency response of the loop gain in the referenced paper

and another component to cancel the effects of the disturbance. To see these components separately, it is only necessary to compute the transform of the control signal in terms of the transforms of the system and of the reference and disturbance as follows.

$$U = \frac{C}{1+PC}R - \frac{PC}{1+PC}W_e$$

As before, if $|PC|$ is large over the bandwidth of the disturbance, the second term in this expression will closely approximate $-W_e$ and can be properly termed as the transform of an estimate of the equivalent disturbance, and marked in the time domain as $\hat{w}_e$. Thus the classical high gain frequency response design can also be seen as a method of equivalent distrubance estimation. In the case of the example given in the paper referenced, the frequency response of PC is given in Fig.10. The example states that the disturbance is restricted in bandwidth to frequencies less than $13 rad/\sec$. From the graph, it can be seen that that the response of the loop has a crossover at $70 rad/\sec$ and a gain of about $17db$ at $13 rad/\sec$. Other than the interesting shape just past crossover, the plot can be interpreted as a classically designed loop for the specified bandwidth.

5 Examples

A set of MATLAB scripts (.m files) are listed in the Appendix to this note to aid in the study of these two alternative approaches to robust design. To illustrate the methods, two designs will be compared for a plant with the transfer function $\frac{1}{s(s+1)}$, typical of a servomechanism. In one design, the system will be designed to track a reference sinusoidal input and in the other case to track a reference step and also reject a disturbance step. In both cases the control signal will be limited by a saturation value selected by the designer. The plant, a sterotypical servomechanism, is described by the equations:

$$\begin{aligned} F &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}; G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ H &= \begin{bmatrix} 1 & 0 \end{bmatrix}; J = 0 \end{aligned}$$

The Internal Model system is designed to have control poles at
-1.0000 + 2.0000i
-1.0000 - 2.0000i
-1.7321 + 1.0000i
-1.7321 - 1.0000i
and plant estimator poles at
-5.0000 + 8.6603i
-5.0000 - 8.6603i
Control poles for the Extended Estimator system are placed at
-1.0000 + 1.7321i
-1.0000 - 1.7321i
and the estimator poles are placed at
-1.7321 + 1.0000i
-1.7321 - 1.0000i
-3.0000 + 5.1962i
-3.0000 - 5.1962i
The choice of these poles is based on a combination of informed guess using the design overshoot and bandwidth. Later experiments would guide the final selection. In this case the results are given by the response curves plotted.

For the case of a system designed to track a sine wave having $\omega = 1$, the outputs are plotted below.

Notice in the next plot that the internal model uses less control and that the control for the Extended Estimator is strongly limited by the saturation.

The errors are plotted below, showing how the Internal Model suffers from using less control

For the system designed to track a step, the outputs, are plotted below.

In this case, the Extended Estimator controls really bounce around as a result of the saturation but the response seems quite reasonable. Finally, the errors indicate comparable results. The results are a bit mixed as the IM design does a bit better tracking the reference while the Extended Estimator rejects the disturbance better.

In the next example, the system that was designed to track a *sine* wave was given a step input. The resulting errors are plotted below. In this case, notice that while the extended estimator can use the implicit internal model of

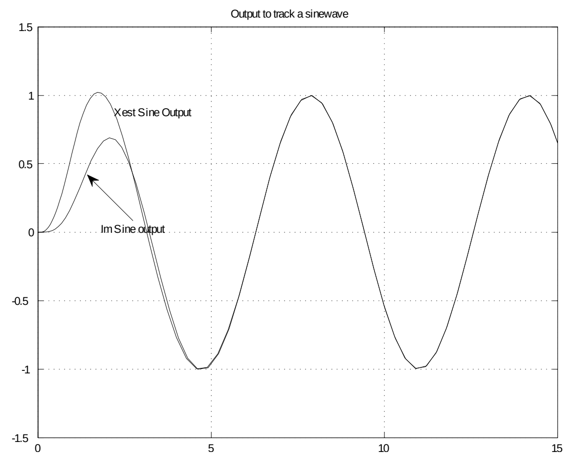

Figure 11: Output responses of the two robust designs to a sine wave

the plant and still gives zero final error, the IM design has a steady state error because the plant estimator has moved the plant pole away from the origin so it is no longer an internal model of the step..

For comparison with the Model Following case, the systems designed to track a sinewave were run but with the plant purturbed to be

$$F = \begin{bmatrix} 0 & 1 \\ 0 & -1.1 \end{bmatrix}$$

The errors are plotted in Fig.18.

Notice that the Model Following design has the smallest maximum error but, being non-robust, has a persistent error while the other two designs continue the track the sine wave exactly.

6 Problems

1. Compute the zeros from the disturbance to the system error for the Internal Model case and explain why the plant pole at the origin does not act as an implicit internal model for step inputs in this case.

2. For the Extended Estimator case, is the plant integrator an implicit internal model or not? Why or why not?.

3. Repeat the IM design with the same plant but take $d(p) = p^3 + p$. You'll need another pole to make the design and you are to place it at $s = -3$. Show that this design for the IM case has an internal model for both $sin(t)$ and $1(t)$ inputs.

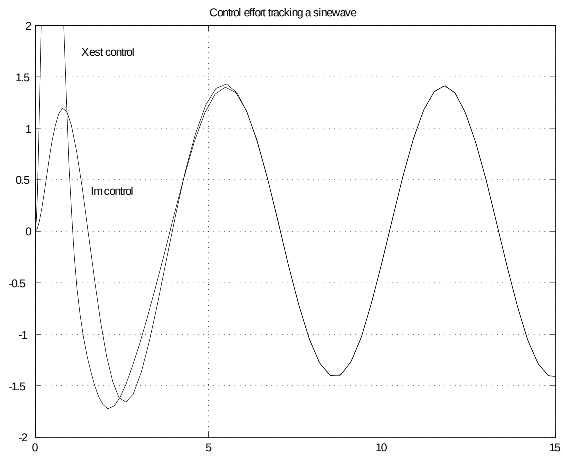

Figure 12: Control signals for the two designs in response to a sine wave

4. Repeat Problem 3 with the Extended Estimator design and compare the results with those of Problem 3 for both step and $\sin(t)$ reference and disturbance.

5. Replace the saturation with a gain, K, and plot the root locus with respect to K for both design methods.. Do these loci suggest that the systems can be driven unstable for very large inputs? Which will do so for the smaller value of K?

7 Appendix Model Following

A related method to track a persistent reference is called Model Following (see .Fig. 19) This is an open loop method that uses the state of the model to construct that particular control input which will force the plant output to asymptotically track the output of the model. The model output may or may not be persistent. The method is described more fully in Bryson[5], including the case of disturbance rejection, and used to synthesize the landing flare logic for the 747 aircraft. The idea is that with a plant described by F, G,H, J having state $\mathbf{x}$ and output y and with a given model described by A,B,C,D with state $\mathbf{z}$ and output η,to use the states $\mathbf{x}$ and $\mathbf{z}$ to construct a control signal so that the error $y - \eta$ 'quickly' approaches zero. Rather than derive the design from first principles, we state the equations and demonstrate that they work. .Consider the plant described by

$$\dot{\mathbf{x}} = F\mathbf{x} + Gu \tag{9}$$

$$y = H\mathbf{x} \tag{10}$$

[5] Control of Spacecraft and Aircraft, A E Bryson, Princeton University Press, 1994

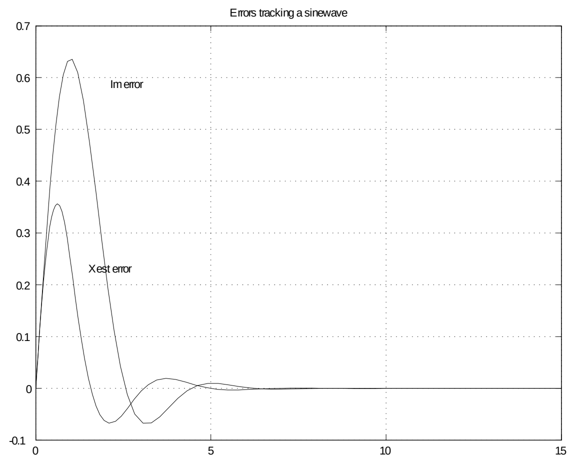

Figure 13: Error signals for the two designs in response to a sine wane

and the model given by

$$\dot{\mathbf{z}} = A\mathbf{z} + B\delta(t) \quad (11)$$

$$\eta = C\mathbf{z} \quad (12)$$

In this case, the model is driven by an impulse or essentially initial conditions only. Let the control be

$$u = N\mathbf{z} - K(\mathbf{x} - M\mathbf{z}) \quad (13)$$

where K is designed in the usual control law way so that $F-GK$ is a satisfactory stable control and the parameters M and N are selected so that

$$FM - MA + GN = 0 \quad (14)$$

$$HM = C \quad (15)$$

With the given control law, Eq.(13) the plant equations become

$$\begin{aligned} \dot{\mathbf{x}} &= F\mathbf{x} + G(N\mathbf{z} - K(\mathbf{x} - M\mathbf{z})) \\ &= (F - GK)\mathbf{x} + (GN + GKM)\mathbf{z} \end{aligned}$$

In the transform domain, noting that $\mathbf{Z}(s) = (sI - A)^{-1}B$, this can be written as

$$\mathbf{X}(s) = (sI - F + GK)^{-1}(GN + GKM)(sI - A)^{-1}B$$

Now substituting for GN from Eq.14 and adding and substracting sM this can be written as

$$\begin{aligned} \mathbf{X}(s) &= (sI - F + GK)^{-1}[MA - FM + GKM](sI - A)^{-1}B \\ \mathbf{X}(s) &= (sI - F + GK)^{-1}[(sI - F + GK)M - M(sI - A)](sI - A)^{-1}B \end{aligned}$$

If we now multiply this out, the result is

$$\mathbf{X}(s) = M(sI - A)^{-1}B - (sI - F + GK)^{-1}MB$$

The output, $Y(s) = H\mathbf{X}(s)$ is thus

$$Y(s) = HM(sI - A)^{-1}B - H(sI - F + GK)^{-1}MB$$

Finally, as $HM = C$, we have

$$Y(s) = C(sI - A)^{-1}B - H(sI - F + GK)^{-1}MB$$

and therefore, in the time domain,

$$y(t) = \eta(t) - [\text{transient term controlled by } K]$$

which was what we set out to show.

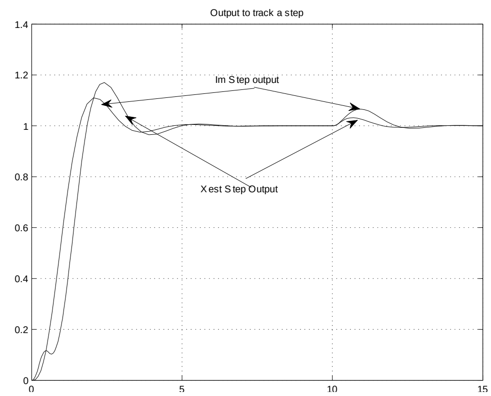

Figure 14: Outputs of the two designs to a step reference

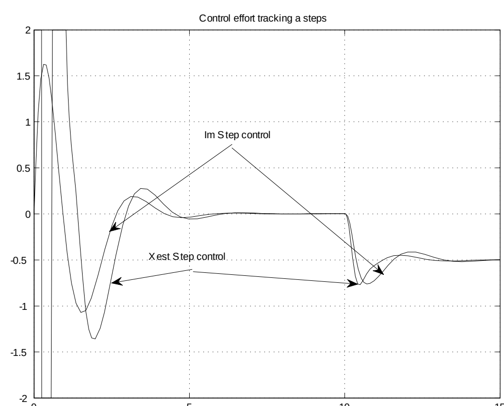

Figure 15: Controls for the two designs to a reference step

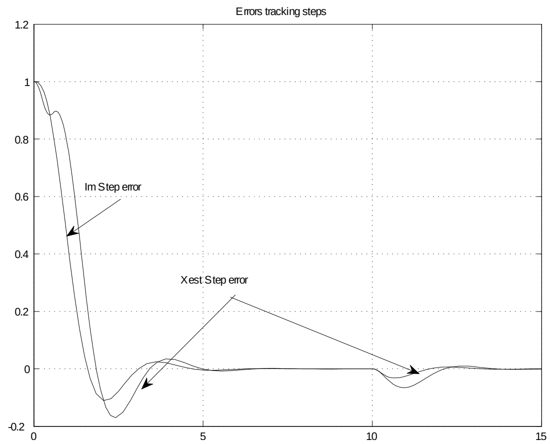

Figure 16: Error signals for the two designs to a reference step

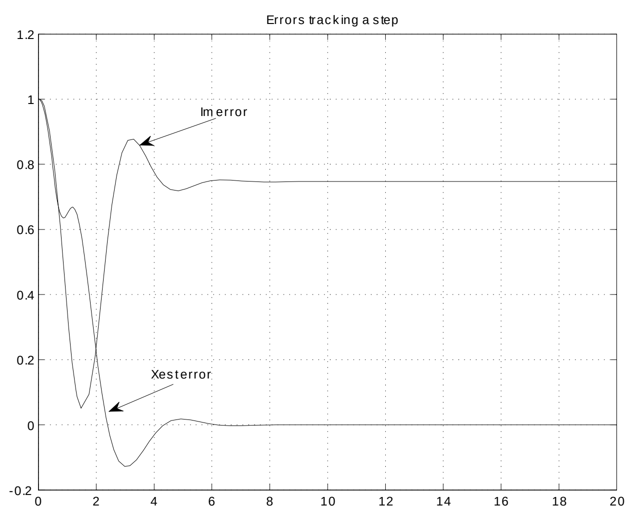

Figure 17: Error responses of the systems designed to track a sine wave when the input is a step

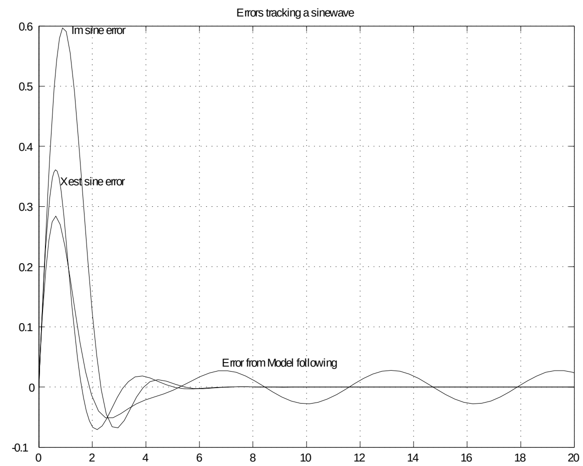

Figure 18: Errors of the three designs with a purturbed plant.

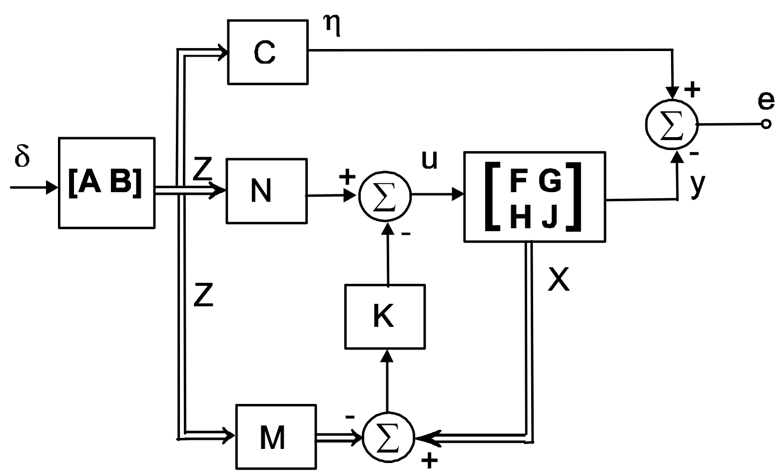

Figure 19: Block Diagram for the Model Following Design