Appendix D

A Review of Complex Variables

This appendix is a brief summary of some results on complex variables theory, with emphasis on the facts needed in control theory. For a comprehensive study of basic complex variables theory, see standard textbooks such as Brown and Churchill (1996) or Marsden and Hoffman (1998).

D.1 Definition of a Complex Number

The complex numbers are distinguished from purely real numbers in that they also contain the **imaginary operator**, which we shall denote j. By definition,

$$j^2 = -1 \quad \text{or} \quad j = \sqrt{-1}. \tag{D.1}$$

A **complex number** may be defined as

$$A = \sigma + j\omega, \tag{D.2}$$

where σ is the real part and ω is the imaginary part, denoted respectively as

$$\sigma = \mathrm{Re}(A), \quad \omega = \mathrm{Im}(A). \tag{D.3}$$

Note that the imaginary part of A is itself a real number.

Graphically, we may represent the complex number A in two ways. In the Cartesian coordinate system (Fig. D.1a), A is represented by a single point in the complex plane. In the polar coordinate system, A is represented by a vector with length r and an angle θ; the angle is measured in radians counterclockwise from the positive real axis (Fig. D.1b). In polar form the complex number A is denoted by

$$A = |A| \cdot \angle \arg A = r \cdot \angle\theta = re^{j\theta}, \quad 0 \le \theta \le 2\pi, \tag{D.4}$$

where r—called the **magnitude**, **modulus**, or **absolute value** of A—is the length of the vector representing A, namely,

$$r = |A| = \sqrt{\sigma^2 + \omega^2}, \tag{D.5}$$

and where θ is given by

$$\tan\theta = \frac{\omega}{\sigma} \tag{D.6}$$

or

$$\theta = \arg(A) = \tan^{-1}\left(\frac{\omega}{\sigma}\right). \tag{D.7}$$

Care must be taken to compute the correct value of the angle, depending on the sign of the real and imaginary parts (i.e., one must find the quadrant in which the complex number lies).

The **conjugate** of A is defined as

$$A^* = \sigma - j\omega\cdot \tag{D.8}$$

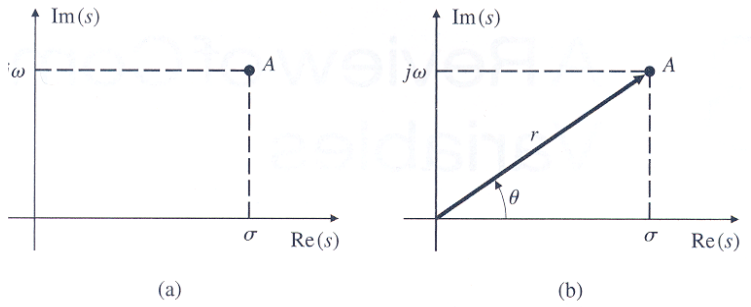

Figure D.1: The complex number A represented in (a) Cartesian and (b) polar coordinates

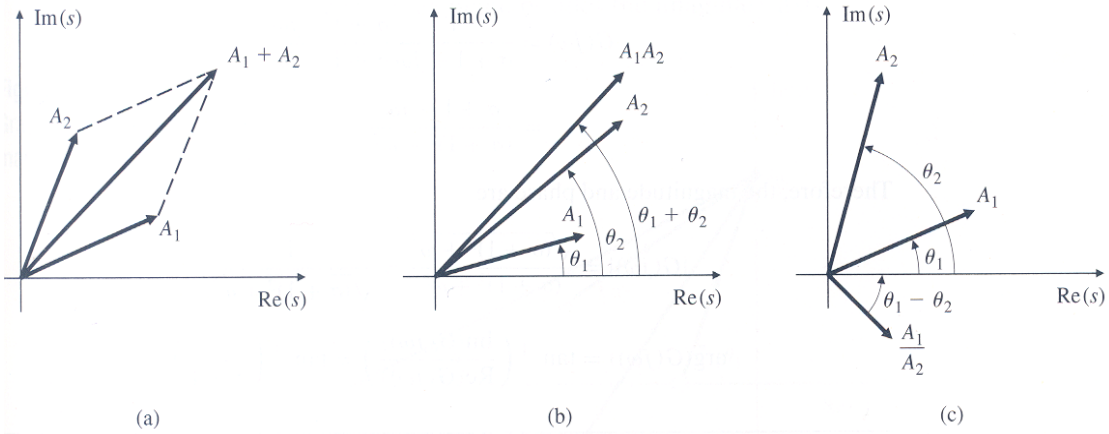

Figure D.2: Arithmetic of complex numbers: (a) addition; (b) multiplication; (c) division

Therefore,

$$(A^*)^* = A, \tag{D.9}$$

$$(A_1 \pm A_2)^* = A_1^* \pm A_2^*, \tag{D.10}$$

$$\left(\frac{A_1}{A_2}\right)^* = \frac{A_1^*}{A_2^*}, \tag{D.11}$$

$$(A_1 A_2)^* = A_1^* A_2^*, \tag{D.12}$$

$$\mathrm{Re}(A) = \frac{A + A^*}{2}, \quad \mathrm{Im}(A) = \frac{A - A^*}{2j}, \tag{D.13}$$

$$AA^* = (|A|)^2. \tag{D.14}$$

D.2 Algebraic Manipulations

D.2.1 Complex Addition

If we let

$$A_1 = \sigma_1 + j\omega_1 \quad \text{and} \quad A_2 = \sigma_2 + j\omega_2, \tag{D.15}$$

then

$$A_1 + A_2 = (\sigma_1 + j\omega_1) + (\sigma_2 + j\omega_2) = (\sigma_1 + \sigma_2) + j(\omega_1 + \omega_2). \tag{D.16}$$

Because each complex number is represented by a vector extending from the origin, we can add or subtract complex numbers graphically. The sum is obtained by adding the two vectors. This we do by constructing a parallelogram and finding its diagonal, as shown in Fig. D.2(a). Alternatively, we could start at the tail of one vector, draw a vector parallel to the other vector, and then connect the origin to the new arrowhead.

Complex subtraction is very similar to complex addition.

D.2.2 Complex Multiplication

For two complex numbers defined according to Eq. (D.15),

$$\begin{aligned} A_1 A_2 &= (\sigma_1 + j\omega_1)(\sigma_2 + j\omega_2) \\ &= (\sigma_1\sigma_2 - \omega_1\omega_2) + j(\omega_1\sigma_2 + \sigma_1\omega_2). \end{aligned} \tag{D.17}$$

The product of two complex numbers may be obtained graphically using polar representations, as shown in Fig. D.2(b).

D.2.3 Complex Division

The division of two complex numbers is carried out by **rationalization**. This means that both the numerator and denominator in the ratio are multiplied by the conjugate of the denominator:

$$\begin{aligned} \frac{A_1}{A_2} &= \frac{A_1 A_2^*}{A_2 A_2^*} \\ &= \frac{(\sigma_1\sigma_2 + \omega_1\omega_2) + j(\omega_1\sigma_2 - \sigma_1\omega_2)}{\sigma_2^2 + \omega_2^2}. \end{aligned} \tag{D.18}$$

From Eq. (D.4) it follows that

$$A^{-1} = \frac{1}{r}e^{-j\theta}, \quad r \neq 0. \tag{D.19}$$

Also, if $A_1 = r_1 e^{j\theta_1}$ and $A_2 = r_2 e^{j\theta_2}$, then

$$A_1 A_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}, \tag{D.20}$$

where $|A_1 A_2| = r_1 r_2$ and $\arg(A_1 A_2) = \theta_1 + \theta_2$, and

$$\frac{A_1}{A_2} = \frac{r_1}{r_2}e^{j(\theta_1 - \theta_2)}, \quad r_2 \neq 0, \tag{D.21}$$

where $\left|\frac{A_1}{A_2}\right| = \frac{r_1}{r_2}$ and $\arg\left(\frac{A_1}{A_2}\right) = \theta_1 - \theta_2$. The division of complex numbers may be carried out graphically in polar coordinates as shown in Fig. D.2(c).

Example D.1 *Frequency Response of First-Order SystemFind the magnitude and phase of the transfer function* $G(s) = \frac{1}{s+1}$, *where* $s = j\omega$.

SOLUTION Substituting $s = j\omega$ *and rationalizing, we obtain*

$$\begin{aligned} G(j\omega) &= \frac{1}{\sigma + 1 + j\omega}\frac{\sigma + 1 - j\omega}{\sigma + 1 - j\omega} \\ &= \frac{\sigma + 1 - j\omega}{(\sigma+1)^2 + \omega^2}. \end{aligned}$$

Therefore, the magnitude and phase are

$$\begin{aligned} |G(j\omega)| &= \frac{\sqrt{(\sigma+1)^2 + \omega^2}}{(\sigma+1)^2 + \omega^2} = \frac{1}{\sqrt{(\sigma+1)^2 + \omega^2}}, \\ \arg(G(j\omega)) &= \tan^{-1}\left(\frac{Im(G(j\omega))}{Re(G(j\omega))}\right) = \tan^{-1}\left(\frac{-\omega}{\sigma+1}\right). \end{aligned}$$

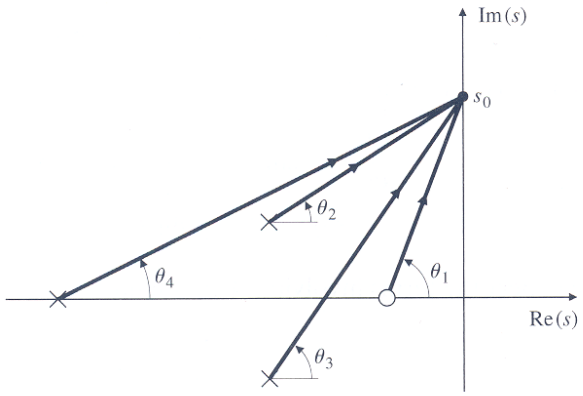

Figure D.3: Graphical determination of magnitude and phase

D.3 Graphical Evaluation of Magnitude and Phase

Consider the transfer function

$$G(s) = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}. \tag{D.22}$$

The value of the transfer function for sinusoidal inputs is found by replacing s with $j\omega$. The gain and phase are given by $G(j\omega)$ and may be determined analytically or by a graphical procedure. Consider the pole-zero configuration for such a $G(s)$ and a point $s_0 = j\omega_0$ on the imaginary axis, as shown in Fig. D.3. Also consider the vectors drawn from the poles and the zero to s_0. The magnitude of the transfer function evaluated at $s_0 = j\omega_0$ is simply the ratio of the distance from the zero to the product of all the distances from the poles:

$$|G(j\omega_0)| = \frac{r_1}{r_2 r_3 r_4}. \tag{D.23}$$

The phase is given by the sum of the angles from the zero minus the sum of the angles from the poles:

$$\arg G(j\omega_0) = \angle G(j\omega_0) = \theta_1 - (\theta_2 + \theta_3 + \theta_4). \tag{D.24}$$

This may be explained as follows. The term $s + z_1$ is a vector addition of its two components. We may determine this equivalently as $s - (-z_1)$, which amounts to translation of the vector $s + z_1$ starting at $-z_1$, as shown in Fig. D.4. This means that a vector drawn from the zero location to s_0 is equivalent to $s + z_1$. The same reasoning applies to the poles. We reflect p_1, p_2, and p_3 about the origin to obtain the pole locations. Then the vectors drawn from $-p_1$, $-p_2$, and $-p_3$ to s_0 are the same as the vectors in the denominator represented in polar coordinates. Note that this method may also be used to evaluate s_0 at places in the complex plane besides the imaginary axis.

D.4 Differentiation and Integration

The usual rules apply to complex differentiation. Let $G(s)$ be differentiable with respect to s. Then the derivative at s_0 is defined as

$$G'(s_0) = \lim_{s \to s_0} \frac{G(s) - G(s_0)}{s - s_0}, \tag{D.25}$$

provided that the limit exists. For conditions on the existence of the derivative, see Brown and Churchill (1996).

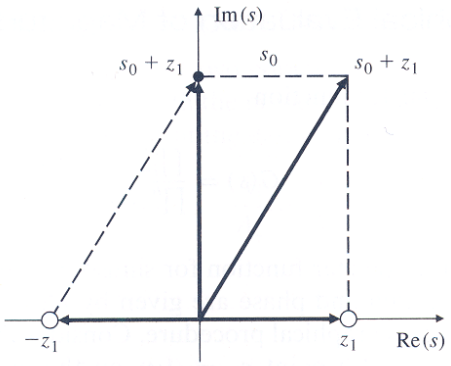

Figure D.4: Illustration of graphical computation of $s + z_1$

The standard rules also apply to integration, except that the constant of integration c is a complex constant:

$$\int G(s)ds = \int \mathrm{Re}[G(s)]ds + j \int \mathrm{Im}[G(s)]ds + c. \tag{D.26}$$

D.5 Euler's Relations

Let us now derive an important relationship involving the complex exponential. If we define

$$A = \cos\theta + j\sin\theta, \tag{D.27}$$

where θ is in radians, then

$$\begin{aligned} \frac{dA}{d\theta} &= -\sin\theta + j\cos\theta = j^2\sin\theta + j\cos\theta \\ &= j(\cos\theta + j\sin\theta) = jA. \end{aligned} \tag{D.28}$$

We collect the terms involving A to obtain

$$\frac{dA}{A} = jd\theta. \tag{D.29}$$

Integrating both sides of Eq. (D.29) yields

$$\ln A = j\theta + c, \tag{D.30}$$

where c is a constant of integration. If we let $\theta = 0$ in Eq. (D.30), we find that $c = 0$ or

$$A = e^{j\theta} = \cos\theta + j\sin\theta. \tag{D.31}$$

Similarly,

$$A^* = e^{-j\theta} = \cos\theta - j\sin\theta. \tag{D.32}$$

From Eqs. (D.31) and (D.32) it follows that Euler's relations

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \tag{D.33}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}. \tag{D.34}$$

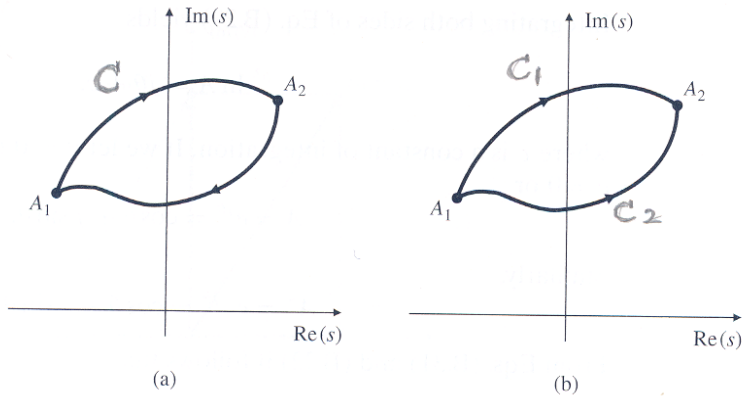

Figure D.5: Contours in the s-plane: (a) a closed contour; (b) two different paths between A_1 and A_2

D.6 Analytic Functions

Let us assume that G is a complex-valued function defined in the complex plane. Let s_0 be in the domain of G, which is assumed to be finite within some disk centered at s_0. Thus, $G(s)$ is defined not only at s_0 but also at all points in the disk centered at s_0. The function G is said to be **analytic** if its derivative exists at s_0 and at each point in the neighborhood of s_0.

D.7 Cauchy's Theorem

A **contour** is a piecewise-smooth arc that consists of a number of smooth arcs joined together. A **simple closed contour** is a contour that does not intersect itself and ends on itself. Let C be a closed contour as shown in Fig. D.5(a), and let G be analytic inside and on C. Cauchy's theorem states that

$$\oint_C G(s)ds = 0. \tag{D.35}$$

There is a corollary to this theorem: Let C_1 and C_2 be two paths connecting the points A_1 and A_2 as in Fig. D.5(b). Then

$$\int_{C_1} G(s)ds = \int_{C_2} G(s)ds. \tag{D.36}$$

D.8 Singularities and Residues

If a function $G(s)$ is not analytic at s_0 but is analytic at some point in every neighborhood of s_0, it is said to be a **singularity**. A singular point is said to be an **isolated singularity** if $G(s)$ is analytic everywhere else in the neighborhood of s_0 except at s_0. Let $G(s)$ be a **rational function** (that is, a ratio of polynomials). If the numerator and denominator are both analytic, then $G(s)$ will be analytic except at the locations of the poles (that is, at roots of the denominator). All singularities of rational algebraic functions are the pole locations.

Let $G(s)$ be analytic except at s_0. Then we may write $G(s)$ in its Laurent series expansion form:

$$G(s) = \frac{A_{-n}}{(s-s_0)^n} + \ldots + \frac{A_{-1}}{(s-s_0)} + B_0 + B_1(s-s_0) + \ldots. \tag{D.37}$$

The coefficient A_{-1} is called the **residue** of $G(s)$ at s_0 and may be evaluated as

$$A_{-1} = \text{Res}[G(s); s_0] = \frac{1}{2\pi j}\oint_C G(s)\ ds, \tag{D.38}$$

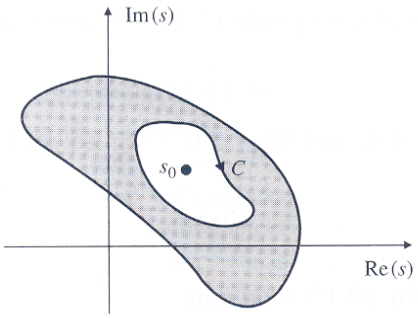

Figure D.6: Contour around an isolated singularity

where C denotes a closed arc within an analytic region centered at s_0 that contains no other singularity, as shown in Fig. D.6. When s_0 is not repeated with $n = 1$, we have

$$A_{-1} = \text{Res}[G(s); s_0] = (s - s_0)G(s)|_{s=s_0}. \tag{D.39}$$

This is the familiar cover-up method of computing residues.

D.9 Residue Theorem

If the contour C contains l singularities, then Eq. (D.39) maybe generalized to yield **Cauchy's residue theorem**:

$$\frac{1}{2\pi j} \oint G(s)\ ds = \sum_{i=1}^{l} \text{Res}[G(s); s_i]. \tag{D.40}$$

D.10 The Argument Principle

Before stating the argument principle, we need a preliminary result from which the principle follows readily.

Number of Poles and Zeros

Let $G(s)$ be an analytic function inside and on a closed contour C, except for a finite number of poles inside C. Then, for C described in the positive sense (clockwise direction),

$$\frac{1}{2\pi j} \oint \frac{G'(s)}{G(s)}\ ds = N - P \tag{D.41}$$

or

$$\frac{1}{2\pi j} \oint d(\ln G) = N - P, \tag{D.42}$$

where N and P are the total number of zeros and poles of G inside C, respectively. A pole or zero of multiplicity k is counted k times.

Proof Let s_0 be a zero of G with multiplicity k. Then, in some neighborhood of that point, we may write $G(s)$ as

$$G(s) = (s - s_0)^k f(s), \tag{D.43}$$

where $f(s)$ is analytic and $f(s_0) \neq 0$. If we differentiate Eq. (D.43), we obtain

$$G'(s) = k(s - s_0)^{k-1} f(s) + (s - s_0)^k f'(s). \tag{D.44}$$

Equation (D.41) may be rewritten as

$$\frac{G'(s)}{G(s)} = \frac{k}{s - s_0} + \frac{f'(s)}{f(s)}. \tag{D.45}$$

Therefore, $G'(s)/G(s)$ has a pole at $s = s_0$ with residue K. This analysis may be repeated for every zero. Hence, the sum of the residues of $G'(s)/G(s)$ is the number of zeros of $G(s)$ inside C. If s_0 is a pole with multiplicity l, we may write

$$h(s) = (s - s_0)^l G(s), \tag{D.46}$$

where $h(s)$ is analytic and $h(s_0) \neq 0$. Then Eq. (D.46) may be rewritten as

$$G(s) = \frac{h(s)}{(s - s_0)^l}. \tag{D.47}$$

Differentiating Eq. (D.47) we obtain

$$G'(s) = \frac{h'(s)}{(s - s_0)^l} - \frac{lh(s)}{(s - s_0)^{l+1}}, \tag{D.48}$$

so that

$$\frac{G'(s)}{G(s)} = \frac{-l}{s - s_0} + \frac{h'(s)}{h(s)}. \tag{D.49}$$

This analysis may be repeated for every pole. The result is that the sum of the residues of $G'(s)/G(s)$ at all the poles of $G(s)$ is $-P$.

The Argument Principle

Using Eq. (D.38), we get

$$\frac{1}{2\pi j} \oint_C d[\ln G(s)] = N - P, \tag{D.50}$$

where $d[\ln G(s)]$ was substituted for $G'(s)/G(s)$. If we write $G(s)$ in polar form, then

$$\begin{aligned} \oint_\Gamma d[\ln G(s)] &= \oint_\Gamma d\{\ln |G(s)| + j \arg[\ln G(s)]\} \\ &= \ln |G(s)||_{s=s_1}^{s=s_2} + j \arg G(s)|_{s=s_1}^{s=s_2}. \end{aligned} \tag{D.51}$$

Because Γ is a closed contour, the first term is zero, but the second term is 2π times the net encirclements of the origin:

$$\frac{1}{2\pi j} \oint_\Gamma d[\ln G(s)] = N - P. \tag{D.52}$$

Intuitively, the argument principle may be stated as follows: We let $G(s)$ be a rational function that is analytic except at possibly a finite number of points. We select an arbitrary contour in the s-plane so that $G(s)$ is analytic at every point on the contour (the contour does not pass through any of the singularities). The corresponding mapping into the $G(s)$-plane may encircle the origin. The number of times it does so is determined by the difference between the number of zeros and the number of poles of $G(s)$ encircled by the s-plane contour. The direction of this encirclement is determined by which is greater, N (clockwise) or P (counterclockwise). For example, if the contour encircles a single zero, the mapping will encircle the origin once in the clockwise direction. Similarly, if the contour encloses only a single pole, the mapping will encircle the origin, this time in the counterclockwise direction. If the contour encircles no singularities, or if the contour encloses an equal number of poles and zeros, there will be no encirclement of the origin. A contour evaluation of $G(s)$ will encircle the origin if there is a nonzero net difference between the encircled singularities. The mapping is **conformal** as well, which means that the magnitude and sense of the angles between smooth arcs is preserved. Chapter 6 provides a more detailed intuitive treatment of the argument principle and its application to feedback control in the form of the Nyquist stability theorem.

D.11 Bilinear Transformation

A bilinear transformation is of the form

$$w = \frac{as + b}{cs + d}, \tag{D.53}$$

where a, b, c, d are complex constants and it is assumed that $ad - bc \neq 0$. The bilinear transformation always transforms circles in the w-plane into circles in the s-plane. This can be shown in several ways. If we solve for s, we obtain

$$s = \frac{-dw + b}{cw - a}. \tag{D.54}$$

The equation for a circle in the w-plane is of the form

$$\frac{|w - \sigma|}{|w - \rho|} = R. \tag{D.55}$$

If we substitute for w in terms of s in Eq. (D.53), we get

$$\frac{|s - \sigma'|}{|s - \rho'|} = R', \tag{D.56}$$

where

$$\sigma' = \frac{\sigma d - b}{a - \sigma c}, \quad \rho' = \frac{\rho d - b}{a - \rho c}, \quad R' = \left|\frac{a - \rho c}{a - \sigma c}\right| R, \tag{D.57}$$

which is the equation for a circle in the s-plane. For alternative proofs the reader is referred to Brown and Churchill (1996) and Marsden and Hoffman (1998).

Appendix E

Summary of Matrix Theory

In the text, we assume you are already somewhat familiar with matrix theory and with the solution of linear systems of equations. However, for the purposes of review we present here a brief summary of matrix theory with an emphasis on the results needed in control theory. For further study, see Strang (1988) and Gantmacher (1959).

E.1 Matrix Definitions

An array of numbers arranged in rows and columns is referred to as a **matrix**. If $\mathbf{A}$ is a matrix with m rows and n columns, an $m \times n$ (read "m by n") matrix, it is denoted by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \tag{E.1}$$

where the entries a_{ij} are its elements. If $m = n$, then the matrix is **square**; otherwise it is **rectangular**. Sometimes a matrix is simply denoted by $\mathbf{A} = [a_{ij}]$. If $m = 1$ or $n = 1$, then the matrix reduces to a **row vector** or a **column vector**, respectively. A **submatrix** of $\mathbf{A}$ is the matrix with certain rows and columns removed.

E.2 Elementary Operations on Matrices

If $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same dimension, then their sum is defined by

$$\mathbf{C} = \mathbf{A} + \mathbf{B}, \tag{E.2}$$

where

$$c_{ij} = a_{ij} + b_{ij}. \tag{E.3}$$

That is, the addition is done element by element. It is easy to verify the following properties of matrices:

Commutative law for addition

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \tag{E.4}$$

Associative law for addition

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}). \tag{E.5}$$

Two matrices can be multiplied if they are compatible. Let $\mathbf{A} = m \times n$ and $\mathbf{B} = n \times p$. Then the $m \times p$ matrix

$$\mathbf{C} = \mathbf{AB} \tag{E.6}$$

is the product of the two matrices, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{E.7}$$

Associative law for multiplication

Matrix multiplication satisfies the associative law

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}, \tag{E.8}$$

but not the commutative law; that is, in general,

$$\mathbf{AB} \neq \mathbf{BA}. \tag{E.9}$$

E.3 Trace

The **trace** of a square matrix is the sum of its diagonal elements:

$$\text{trace } \mathbf{A} = \sum_{i=1}^{n} a_{ii}. \tag{E.10}$$

E.4 Transpose

The $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$ is called the **transpose of matrix A**:

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}.$$

A matrix is said to be **symmetric** if

$$\mathbf{A}^T = \mathbf{A}. \tag{E.11}$$

Transposition

It is easy to show that

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{E.12}$$

$$(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T \tag{E.13}$$

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T. \tag{E.14}$$

E.5 Determinant and Matrix Inverse

The **determinant** of a square matrix is defined by Laplace's expansion

$$\det \mathbf{A} = \sum_{j=1}^{n} a_{ij}\gamma_{ij} \quad \text{for any} \quad i = 1, 2, \dots, n, \tag{E.15}$$

where γ_{ij} is called the **cofactor** and

$$\gamma_{ij} = (-1)^{i+j} \det M_{ij}, \tag{E.16}$$

where the scalar $\det M_{ij}$ is called a **minor**. M_{ij} is the same as the matrix $\mathbf{A}$ except that its ith row and jth column have been removed. Note that M_{ij} is always an $(n-1) \times (n-1)$ matrix, and that the minors and cofactors are identical except possibly for a sign.

The **adjugate** of a matrix is the transpose of the matrix of its cofactors:

$$\text{adj } \mathbf{A} = [\gamma_{ij}]^T. \tag{E.17}$$

It can be shown that

$$\mathbf{A} \text{ adj } \mathbf{A} = (\det \mathbf{A})\mathbf{I}, \tag{E.18}$$

natrix where $\mathbf{I}$ is called the **identity matrix**:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix};$$

that is, $\mathbf{I}$ has ones along the diagonal and zeros elsewhere. If $\det \mathbf{A} \neq 0$, then the **inverse** of a matrix $\mathbf{A}$ is defined by

$$\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{\det \mathbf{A}} \tag{E.19}$$

and has the property

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}. \tag{E.20}$$

Note that a matrix has an inverse—that is, it is **nonsingular**—if its determinant is nonzero.

The inverse of the product of two matrices is the product of the inverse of the matrices in reverse order:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{E.21}$$

and Inversion

$$(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}. \tag{E.22}$$

E.6 Properties of the Determinant

When dealing with determinants of matrices, the following elementary (row or column) operations are useful:

1. If any row (or column) of $\mathbf{A}$ is multiplied by a scalar α, the resulting matrix $\bar{\mathbf{A}}$ has the determinant

   $$\det \bar{\mathbf{A}} = \alpha \det \mathbf{A}. \tag{E.23}$$

   Hence

   $$\det(\alpha \mathbf{A}) = \alpha^n \det \mathbf{A}. \tag{E.24}$$

2. If any two rows (or columns) of $\mathbf{A}$ are interchanged to obtain $\bar{\mathbf{A}}$, then

   $$\det \bar{\mathbf{A}} = -\det \mathbf{A}. \tag{E.25}$$

3. If a multiple of a row (or column) of $\mathbf{A}$ is added to another to obtain $\bar{\mathbf{A}}$, then

   $$\det \bar{\mathbf{A}} = \det \mathbf{A}. \tag{E.26}$$

4. It is also easy to show that

   $$\det \mathbf{A} = \det \mathbf{A}^T \tag{E.27}$$

   and

   $$\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}. \tag{E.28}$$

Applying Eq. (E.28) to Eq. (E.20), we have

$$\det \mathbf{A} \det \mathbf{A}^{-1} = 1. \tag{E.29}$$

If $\mathbf{A}$ and $\mathbf{B}$ are square matrices, then the determinant of the block triangular matrix is the product of the determinants of the diagonal blocks:

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ 0 & \mathbf{B} \end{bmatrix} = \det \mathbf{A} \det \mathbf{B}. \tag{E.30}$$

If $\mathbf{A}$ is nonsingular, then

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det \mathbf{A} \det(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}). \tag{E.31}$$

Using this identity, we can write the transfer function of a scalar system in a compact form:

$$G(s) = \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} + J = \frac{\det \begin{bmatrix} (s\mathbf{I} - \mathbf{F}) & \mathbf{G} \\ -\mathbf{H} & J \end{bmatrix}}{\det(s\mathbf{I} - \mathbf{F})}. \tag{E.32}$$

E.7 Inverse of Block Triangular Matrices

If $\mathbf{A}$ and $\mathbf{B}$ are square invertible matrices, then

$$\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix}. \tag{E.33}$$

E.8 Special Matrices

Some matrices have special structures and are given names. We have already defined the identity matrix, which has a special form. A **diagonal matrix** has (possibly) nonzero elements along the main diagonal and zeros elsewhere:

Diagonal matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & & \mathbf{0} & & \\ & a_{22} & & & \\ & & a_{33} & & \\ & & & \ddots & \\ & \mathbf{0} & & & a_{nn} \end{bmatrix}. \tag{E.34}$$

A matrix is said to be **(upper) triangular** if all the elements below the main diagonal are zeros:

Upper triangular matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & & \cdots & a_{1n} \\ 0 & a_{22} & & & \\ \vdots & 0 & & & \vdots \\ 0 & \vdots & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}. \tag{E.35}$$

The determinant of a diagonal or triangular matrix is simply the product of its diagonal elements.

A matrix is said to be in the **(upper) companion form** if it has the structure

$$\mathbf{A}_c = \begin{bmatrix} -a_1 & -a_2 & & \cdots & -a_n \\ 1 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix}. \tag{E.36}$$

Note that all the information is contained in the first row. Variants of this form are the lower, left, or right companion matrices. A **Vandermonde matrix** has the following structure:

$$\mathbf{A} = \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix}. \tag{E.37}$$

E.9 Rank

The **rank** of a matrix is the number of its linearly independent rows or columns. If the rank of $\mathbf{A}$ is r, then all $(r+1) \times (r+1)$ submatrices of $\mathbf{A}$ are singular, and there is at least one $r \times r$ submatrix that is nonsingular. It is also true that

$$\text{row rank of } \mathbf{A} = \text{column rank of } \mathbf{A}. \tag{E.38}$$

E.10 Characteristic Polynomial

The **characteristic polynomial** of a matrix $\mathbf{A}$ is defined by

$$\begin{aligned} a(s) &\triangleq \det(s\mathbf{I} - \mathbf{A}) \\ &= s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n, \end{aligned} \tag{E.39}$$

where the roots of the polynomial are referred to as **eigenvalues** of $\mathbf{A}$. We can write

$$a(s) = (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n), \tag{E.40}$$

where $\{\lambda_i\}$ are the eigenvalues of $\mathbf{A}$. The characteristic polynomial of a companion matrix (e.g., Eq. (E.36)) is

$$\begin{aligned} a(s) &= \det(s\mathbf{I} - \mathbf{A}_c) \\ &= s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n. \end{aligned} \tag{E.41}$$

E.11 Cayley–Hamilton Theorem

The Cayley–Hamilton theorem states that every square matrix $\mathbf{A}$ satisfies its characteristic polynomial. This means that if $\mathbf{A}$ is an $n \times n$ matrix with characteristic equation $a(s)$, then

$$a(\mathbf{A}) \triangleq \mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = 0 \tag{E.42}$$

E.12 Eigenvalues and Eigenvectors

Any scalar λ and nonzero vector $\mathbf{v}$ that satisfy

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{E.43}$$

are referred to as the eigenvalue and the associated **(right) eigenvector** of the matrix $\mathbf{A}$ [because $\mathbf{v}$ appears to the right of $\mathbf{A}$ in Eq. (E.43)]. By rearranging terms in Eq. (E.43), we get

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = 0. \tag{E.44}$$

Because $\mathbf{v}$ is nonzero,

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0, \tag{E.45}$$

so λ is an eigenvalue of the matrix $\mathbf{A}$ as defined in Eq. (E.43). The normalization of the eigenvectors is arbitrary; that is, if $\mathbf{v}$ is an eigenvector, so is $\alpha\mathbf{v}$. The eigenvectors are usually normalized to have unit length; that is, $\|\mathbf{v}\|^2 = \mathbf{v}^T\mathbf{v} = 1$.

If $\mathbf{w}^T$ is a nonzero row vector such that

$$\mathbf{w}^T\mathbf{A} = \lambda\mathbf{w}^T, \tag{E.46}$$

then $\mathbf{w}$ is called a **left eigenvector** of $\mathbf{A}$ [because $\mathbf{w}^T$ appears to the left of $\mathbf{A}$ in Eq. (E.46)]. Note that we can write

$$\mathbf{A}^T\mathbf{w} = \lambda\mathbf{w} \tag{E.47}$$

so that $\mathbf{w}$ is simply a right eigenvector of $\mathbf{A}^T$.

E.13 Similarity Transformations

Consider the arbitrary nonsingular matrix $\mathbf{T}$ such that

$$\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}. \tag{E.48}$$

The matrix operation shown in Eq. (E.48) is referred to as a **similarity transformation**. If $\mathbf{A}$ has a full set of eigenvectors, then we can choose $\mathbf{T}$ to be the set of eigenvectors and $\bar{\mathbf{A}}$ will be diagonal.

Consider the set of equations in state-variable form:

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u. \tag{E.49}$$

If we let

$$\mathbf{T}\boldsymbol{\xi} = \mathbf{x}, \tag{E.50}$$

then Eq. (E.49) becomes

$$\mathbf{T}\dot{\boldsymbol{\xi}} = \mathbf{F}\mathbf{T}\boldsymbol{\xi} + \mathbf{G}u, \tag{E.51}$$

and premultiplying both sides by $\mathbf{T}^{-1}$, we get

$$\begin{aligned} \dot{\xi} &= \mathbf{T}^{-1}\mathbf{F}\mathbf{T}\boldsymbol{\xi} + \mathbf{T}^{-1}\mathbf{G}u \\ &= \bar{\mathbf{F}}\xi + \bar{\mathbf{G}}u, \end{aligned} \tag{E.52}$$

where

$$\begin{aligned} \bar{\mathbf{F}} &= \mathbf{T}^{-1}\mathbf{F}\mathbf{T}, \\ \bar{\mathbf{G}} &= \mathbf{T}^{-1}\mathbf{G}. \end{aligned} \tag{E.53}$$

The characteristic polynomial of $\bar{\mathbf{F}}$ is

$$\begin{aligned} \det(s\mathbf{I} - \bar{\mathbf{F}}) &= \det(s\mathbf{I} - \mathbf{T}^{-1}\mathbf{F}\mathbf{T}) \\ &= \det(s\mathbf{T}^{-1}\mathbf{T} - \mathbf{T}^{-1}\mathbf{F}\mathbf{T}) \\ &= \det[\mathbf{T}^{-1}(s\mathbf{I} - \mathbf{F})\mathbf{T}] \\ &= \det\mathbf{T}^{-1}\det(s\mathbf{I} - \mathbf{F})\det\mathbf{T}. \end{aligned} \tag{E.54}$$

Using Eq. (E.29), Eq. (E.54) becomes

$$\det(s\mathbf{I} - \bar{\mathbf{F}}) = \det(s\mathbf{I} - \mathbf{F}). \tag{E.55}$$

From Eq. (E.55) we can see that $\bar{\mathbf{F}}$ and $\mathbf{F}$ both have the same characteristic polynomial, giving us the important result that a similarity transformation does not change the eigenvalues of a matrix. From Eq. (E.50) a new state made up of a linear combination from the old state has the same eigenvalues as the old set.

E.14 Matrix Exponential

Let $\mathbf{A}$ be a square matrix. The **matrix exponential** of $\mathbf{A}$ is defined as the series

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{\mathbf{A}^3t^3}{3!} + \cdots \tag{E.56}$$

It can be shown that the series converges. If $\mathbf{A}$ is an $n \times n$ matrix, then $e^{\mathbf{A}t}$ is also an $n \times n$ matrix and can be differentiated:

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}. \tag{E.57}$$

Other properties of the matrix exponential are

$$e^{\mathbf{A}t_1}e^{\mathbf{A}t_2} = e^{\mathbf{A}(t_1+t_2)} \tag{E.58}$$

and, in general,

$$e^{\mathbf{A}}e^{\mathbf{B}} \neq e^{\mathbf{B}}e^{\mathbf{A}}. \tag{E.59}$$

(In the exceptional case where $\mathbf{A}$ and $\mathbf{B}$ commute—that is, $\mathbf{AB} = \mathbf{BA}$—then $e^{\mathbf{A}}e^{\mathbf{B}} = e^{\mathbf{B}}e^{\mathbf{A}}$.)

E.15 Fundamental Subspaces

The **range space** of $\mathbf{A}$, denoted by $\mathcal{R}(\mathbf{A})$ and also called the **column space** of $\mathbf{A}$, is defined by the set of vectors

$$\mathbf{x} = \mathbf{A}\mathbf{y} \tag{E.60}$$

for some vector $\mathbf{y}$. The **null space** of $\mathbf{A}$, denoted by $\mathcal{N}(\mathbf{A})$, is defined by the set of vectors $\mathbf{x}$ such that

$$\mathbf{A}\mathbf{x} = \mathbf{0}. \tag{E.61}$$

If $x \in \mathcal{N}(\mathbf{A})$ and $y \in \mathcal{R}(\mathbf{A}^T)$, then $\mathbf{y}^T\mathbf{x} = 0$; that is, every vector in the null space of $\mathbf{A}$ is **orthogonal** to every vector in the range space of $\mathbf{A}^T$.

E.16 Singular-Value Decomposition

The **singular-value decomposition (SVD)** is one of the most useful tools in linear algebra and has been widely used in control theory during the last two decades. Let $\mathbf{A}$ be an $m \times n$ matrix. Then there always exist matrices $\mathbf{U}$, $\mathbf{S}$, and $\mathbf{V}$ such that

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T. \tag{E.62}$$

Here $\mathbf{U}$ and $\mathbf{V}$ are **orthogonal matrices**; that is

$$\mathbf{U}\mathbf{U}^T = \mathbf{I}, \mathbf{V}\mathbf{V}^T = \mathbf{I}. \tag{E.63}$$

$\mathbf{S}$ is a **quasidiagonal matrix** with singular values as its diagonal elements; that is,

$$\mathbf{S} = \begin{bmatrix} \mathbf{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \tag{E.64}$$

where $\mathbf{\Sigma}$ is a diagonal matrix of nonzero singular values in descending order:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0. \tag{E.65}$$

The unique diagonal elements of **S** are called the **singular values**. The maximum singular value is denoted by $\bar{\sigma}(A)$, and the minimum singular value is denoted by $\underline{\sigma}(\mathbf{A})$. The rank of the matrix is the same as the number of nonzero singular values. The columns of **U** and **V**,

$$\begin{aligned} \mathbf{U} &= [\ u_1 \quad u_2 \quad \ldots \quad u_m\ ], \\ \mathbf{V} &= [\ v_1 \quad v_2 \quad \ldots \quad v_n\ ], \end{aligned} \tag{E.66}$$

are called the left and right **singular vectors**, respectively. SVD provides complete information about the fundamental subspaces associated with a matrix:

$$\begin{aligned} \mathcal{N}(\mathbf{A}) &= \text{span}[\ v_{r+1} \quad v_{r+2} \quad \ldots \quad v_n\ ], \\ \mathcal{R}(\mathbf{A}) &= \text{span}[\ u_1 \quad u_2 \quad \ldots \quad u_r\ ], \\ \mathcal{R}(\mathbf{A}^T) &= \text{span}[\ v_1 \quad v_2 \quad \ldots \quad v_r\ ], \\ \mathcal{N}(\mathbf{A}^T) &= \text{span}[\ u_{r+1} \quad u_{r+2} \quad \ldots \quad u_m\ ]. \end{aligned} \tag{E.67}$$

Here $\mathcal{R}$ denotes the null space, and $\mathcal{N}$ denotes the range space respectively.

The **norm** of the matrix **A**, denoted by $\|\mathbf{A}\|_2$, is given by

$$\|\mathbf{A}\|_2 = \bar{\sigma}(\mathbf{A}). \tag{E.68}$$

If **A** is a function of ω, then the infinity norm of **A**, $\|\mathbf{A}\|_\infty$, is given by

$$\|\mathbf{A}(j\omega)\|_\infty = \max_\omega \bar{\sigma}(\mathbf{A}). \tag{E.69}$$

E.17 Positive Definite Matrices

A matrix **A** is said to be **positive semidefinite** if

$$\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0 \quad \text{for all } \mathbf{x}. \tag{E.70}$$

The matrix is said to be **positive definite** if equality holds in Eq. (E.70) only for $\mathbf{x} = 0$. A symmetric matrix is positive definite if and only if all of its eigenvalues are positive. It is positive semidefinite if and only if all of its eigenvalues are nonnegative.

An alternate method for determining positive definiteness is to test the minors of the matrix. A matrix is positive definite if all the leading principal minors are positive, and positive semidefinite if they are all nonnegative.

E.18 Matrix Identity

If **A** is an $n \times m$ matrix and **B** is an $m \times n$ matrix, then

$$\det[\mathbf{I}_n - \mathbf{A}\mathbf{B}] = \det[\mathbf{I}_m - \mathbf{B}\mathbf{A}],$$

where $\mathbf{I}_n$ and $\mathbf{I}_m$ are identity matrices of size n and m respectively.

AppendixwF

Controllability and Observability

Controllability and observability are important structural properties of dynamic systems. First identified and studied by Kalman (1960) and later by Kalman et al. (1961), these properties have continued to be examined during the last four decades. We will discuss only a few of the known results for linear constant systems with one input and one output. In the text we discuss these concepts in connection with control law and estimator designs. For example, in Section 7.4 we suggest that if the square matrix given by

$$\mathcal{C} = [\mathbf{G}\ \mathbf{FG}\ \mathbf{F}^2\ \mathbf{G}\ \ldots\ \mathbf{F}^{n-1}\mathbf{G}] \tag{F.1}$$

is nonsingular, then by transformation of the state we can convert the given description into control canonical form. We can then construct a control law that will give the closed-loop system an arbitrary characteristic equation.

F.1 Controllability

We begin our formal discussion of controllability with the first of four definitions:

Definition 1 *Definition I The system (**F**, **G**) is **controllable** if, for any given nth-order polynomial $\alpha_c(s)$, there exists a (unique) control law $u = -\mathbf{Kx}$ such that the characteristic polynomial of $\mathbf{F}-\mathbf{GK}$ is $\alpha_c(s)$.*

From the results of Ackermann's formula (see Appendix G), we have the following mathematical test for controllability: (**F, G**) is a controllable pair if and only if the rank of $\mathcal{C}$ is n. Definition I based on pole placement is a frequency-domain concept. Controllability can be equivalently defined in the time domain.

Definition 2 *Definition II The system (**F**, **G**) is **controllable** if there exists a (piecewise continuous) control signal $u(t)$ that will take the state of the system from any initial state $\mathbf{x}_0$ to any desired final state $\mathbf{x}_f$ in a finite time interval.*

We will now show that the system is controllable by this definition if and only if $\mathcal{C}$ is full rank. We first assume that the system is controllable but that

$$\text{rank}[\mathbf{G}\ \mathbf{FG}\ \mathbf{F}^2\mathbf{G}\ \ldots\ \mathbf{F}^{n-1}\mathbf{G}] < n. \tag{F.2}$$

We can then find a vector $\mathbf{v}$ such that

$$\mathbf{v}[\mathbf{G}\ \mathbf{FG}\ \mathbf{F}^2\mathbf{G}\ \ldots\ \mathbf{F}^{n-1}\mathbf{G}] = 0, \tag{F.3}$$

or

$$\mathbf{vG} = \mathbf{vFG} = \mathbf{vF}^2\mathbf{G} = \ldots = \mathbf{vF}^{n-1}\mathbf{G} = 0. \tag{F.4}$$

The Cayley–Hamilton theorem states that $\mathbf{F}$ satisfies its own characteristic equation, namely,

$$-\mathbf{F}^n = a_1\mathbf{F}^{n-1} + a_2\mathbf{F}^{n-2} + \ldots + a_n\mathbf{I}. \tag{F.5}$$

Therefore,

$$-\mathbf{v}\mathbf{F}^n\mathbf{G} = a_1\mathbf{v}\mathbf{F}^{n-1}\mathbf{G} + a_2\mathbf{v}\mathbf{F}^{n-2}\mathbf{G} + \ldots + a_n\mathbf{v}\mathbf{G} = 0. \tag{F.6}$$

By induction, $\mathbf{v}\mathbf{F}^{n+k}\mathbf{G} = 0$ for $k = 0, 1, 2, \ldots$, or $\mathbf{v}\mathbf{F}^m\mathbf{G} = 0$ for $m = 0, 1, 2, \ldots$, and thus

$$\mathbf{v}e^{\mathrm{F}t}\mathbf{G} = \mathbf{v}\left(\mathbf{I} + \mathbf{F}t + \frac{1}{2!}\mathbf{F}^2t^2 + \ldots\right)\mathbf{G} = 0 \tag{F.7}$$

for all t. However, the zero initial-condition response ($\mathbf{x}_0 = \mathbf{0}$) is

$$\begin{aligned}\mathbf{x}(t) &= \int_0^t \mathbf{v}e^{\mathbf{F}(t-\tau)}\mathbf{G}u(\tau)\,d\tau \\ &= e^{\mathbf{F}t}\int_0^t e^{-\mathbf{F}t}\mathbf{G}u(\tau)\,d\tau.\end{aligned} \tag{F.8}$$

Using Eq. (F.7), Eq. (F.8) becomes

$$\mathbf{v}\mathbf{x}(t) = \int_0^t \mathbf{v}e^{\mathbf{F}(t-\tau)}\mathbf{G}u(\tau)\,d\tau = 0 \tag{F.9}$$

for all $u(t)$ and $t > 0$. This implies that all points reachable from the origin are orthogonal to $\mathbf{v}$. This restricts the reachable space and therefore contradicts the second definition of controllability. Thus if $\mathcal{C}$ is singular, ($\mathbf{F}$, $\mathbf{G}$) is not controllable by Definition II.

Next we assume that $\mathcal{C}$ is full rank but ($\mathbf{F}$, $\mathbf{G}$) is uncontrollable by Definition II. This means that there exists a nonzero vector $\mathbf{v}$ such that

$$\mathbf{v}\int_0^{t_f} e^{\mathbf{F}(t_f-\tau)}\mathbf{G}u(\tau)\,d\tau = 0, \tag{F.10}$$

because the whole state space is not reachable. But Eq. (F.10) implies that

$$\mathbf{v}e^{\mathbf{F}^{(t_f-\tau)}}\mathbf{G} = 0, \quad 0 \le \tau \le t_f. \tag{F.11}$$

If we set $\tau = t_f$, we see that $\mathbf{v}\mathbf{G} = 0$. Also, differentiating Eq. (F.11) and letting $\tau = t_f$ gives $\mathbf{v}\mathbf{F}\mathbf{G} = 0$. Continuing this process, we find that

$$\mathbf{v}\mathbf{G} = \mathbf{v}\mathbf{F}\mathbf{G} = \mathbf{v}\mathbf{F}^2\mathbf{G} = \ldots = \mathbf{v}\mathbf{F}^{n-1}\mathbf{G} = 0, \tag{F.12}$$

which contradicts the assumption that $\mathcal{C}$ is full rank.

We have now shown that the system is controllable by Definition II if and only if the rank of $\mathcal{C}$ is n, exactly the same condition we found for pole assignment.

Our final definition comes closest to the structural character of controllability:

Definition 3 *Definition III The system (**F**, **G**) is **controllable** if every mode of **F** is connected to the control input.*

Because of the generality of the modal structure of systems, we will treat only the case of systems for which $\mathbf{F}$ can be transformed to diagonal form. (The double-integration plant does *not* qualify.) Suppose we have a diagonal matrix $\mathbf{F}_d$ and its corresponding input matrix $\mathbf{G}_d$ with elements g_i. The structure of such a system is shown in Fig. (F.1). By definition, for a controllable system the input must be connected to each mode so that the

g_i are all nonzero. However, this is not enough if the poles (λ_i) are not distinct. Suppose, for instance, that $\lambda_1 = \lambda_2$. The first two state equations are then

$$\begin{aligned}\dot{x}_{1d} &= \lambda_1 x_{1d} + g_1 u, \\ \dot{x}_{2d} &= \lambda_1 x_{2d} + g_2 u.\end{aligned} \tag{F.13}$$

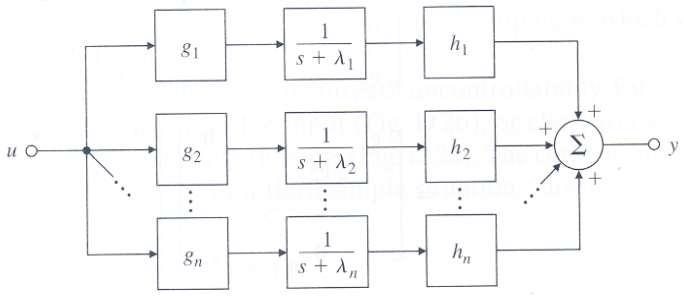

Figure F.1: Block diagram of a system with a diagonal matrix

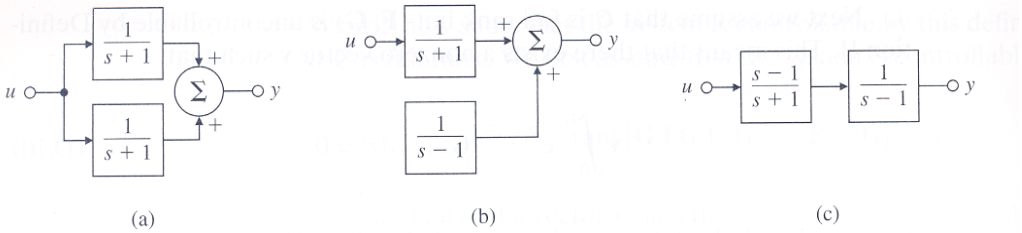

Figure F.2: Examples of uncontrollable systems

If we define a new state, $\xi = g_2 x_{1d} - g_1 x_{2d}$, the equation for ξ is

$$\dot{\xi} = g_2 \dot{x}_{1d} - g_1 \dot{x}_{2d} = g_2 \lambda_1 x_{1d} + g_2 g_1 u - g_1 \lambda_1 x_{2d} - g_1 g_2 u = \lambda_1 \xi, \tag{F.14}$$

which does not include the control u; hence, ξ is not controllable. The point is that if any two poles are equal *in a diagonal* $\mathbf{F}_d$ *system with only one input*, we effectively have a hidden mode that is not connected to the control, and the system is not controllable (Fig. F.2a). This is because the two state variables move together exactly, so we cannot *independently* control x_{1d} and x_{2d}. Therefore, even in such a simple case, we have two conditions for controllability:

1. All eigenvalues of $\mathbf{F}_d$ are distinct.
2. No element of $\mathbf{G}_d$ is zero.

Now let us consider the controllability matrix of this diagonal system. By direct computation,

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} g_1 & g_1\lambda_1 & \dots & g_1\lambda_1^{n-1} \\ g_2 & g_2\lambda_2 & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ g_n & g_n\lambda_n & \dots & g_n\lambda_n^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} g_1 & & & \mathbf{0} \\ & g_2 & & \\ & & \ddots & \\ \mathbf{0} & & & g_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix}. \end{aligned} \tag{F.15}$$

Note that the controllability matrix $\mathcal{C}$ is the product of two matrices and is nonsingular if and only if both of these matrices are invertible. The first matrix has a determinant that is the product of

the g_i, and the second matrix (called a Vandermonde matrix) is nonsingular if and only if the λ_i are distinct. Thus Definition III is equivalent to having a nonsingular $\mathcal{C}$ also.

Important to the subject of controllability is the **Popov–Hautus–Rosenbrock (PHR) test** (see Rosenbrock, 1970, and Kailath, 1980), which is an alternate way to test the rank (or determinant) of $\mathcal{C}$. The system ($\mathbf{F}$, $\mathbf{G}$) is controllable if the system of equations

$$\mathbf{v}^T[s\mathbf{I}-\mathbf{F} \quad \mathbf{G}] = \mathbf{0}^T \tag{F.16}$$

has only the trivial solution $\mathbf{v}^T = \mathbf{0}^T$—that is, if the **matrix pencil**

$$\text{rank}[s\mathbf{I}-\mathbf{F} \quad \mathbf{G}] = n \tag{F.17}$$

is full rank for all s, or if there is no nonzero $\mathbf{v}^T$ such that[1]

$$\mathbf{v}^T\mathbf{F} = s\mathbf{v}^T, \tag{F.18}$$

$$\mathbf{v}^T\mathbf{G} = 0. \tag{F.19}$$

This test is equivalent to the rank-of-$\mathcal{C}$ test. It is easy to show that if such a vector $\mathbf{v}$ exists, then $\mathcal{C}$ is singular. For, if a nonzero $\mathbf{v}$ exists such that $\mathbf{v}^T\mathbf{G} = 0$ then by Eqs. (F.18) and (F.19),

$$\mathbf{v}^T\mathbf{F}\mathbf{G} = s\mathbf{v}^T\mathbf{G} = 0. \tag{F.20}$$

Then, multiplying by $\mathbf{FG}$, we find that

$$\mathbf{v}^T\mathbf{F}^2\mathbf{G} = s\mathbf{v}^T\mathbf{F}\mathbf{G} = 0, \tag{F.21}$$

and so on. Thus we determine that $\mathbf{v}^T\mathcal{C} = \mathbf{0}^T$ has a nontrivial solution, that $\mathcal{C}$ is singular, and that the system is not controllable. To show that a nontrivial $\mathbf{v}^T$ exists if $\mathcal{C}$ is singular requires more development, which we will not give here (see Kailath, 1980).

We have given two pictures of uncontrollability. Either a mode is physically disconnected from the input (Fig. F.2b), or else two parallel subsystems have identical characteristic roots (Fig. F.2a). The control engineer should be aware of the existence of a third simple situation, illustrated in Fig. F.2c, namely, a **pole-zero cancellation**. Here the problem is that the mode at $s = 1$ appears to be connected to the input but is masked by the zero at $s = 1$ in the preceding subsystem; the result is an uncontrollable system. This can be confirmed in several ways. First let us look at the controllability matrix. The system matrices are

$$\mathbf{F} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

so the controllability matrix is

$$\mathcal{C} = [\ \mathbf{G} \quad \mathbf{FG}\ ] = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \tag{F.22}$$

which is clearly singular. The controllability matrix may be computed using the ctrb command in MATLAB: [cc]=ctrb(F,G). If we compute the transfer function from u to x_2, we find that

$$\mathbf{H}(s) = \frac{s-1}{s+1}\left(\frac{1}{s-1}\right) = \frac{1}{s+1}. \tag{F.23}$$

Because the natural mode at $s = 1$ disappears from the input–output description, it is not connected to the input. Finally, if we consider the **PHR** test,

$$[s\mathbf{I}-\mathbf{F} \quad \mathbf{G}] = \begin{bmatrix} s+1 & 0 & -2 \\ -1 & s-1 & 1 \end{bmatrix}, \tag{F.24}$$

and let $s = 1$, then we must test the rank of

$$\begin{bmatrix} 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix},$$

which is clearly less than 2. This result means, again, that the system is uncontrollable.

[1] $\mathbf{v}^T$ is a left eigenvector of $\mathbf{F}$.

Definition 4 *Definition IV The asymptotically stable system (**F**, **G**) is **controllable** if the controllability Gramian, the square symmetric matrix $\mathcal{C}_g$, given by the solution to the Lyapunov equation*

$$\mathbf{F}\mathcal{C}_g + \mathcal{C}_g\mathbf{F}^T + \mathbf{G}\mathbf{G}^T = \mathbf{0}, \tag{F.25}$$

*is nonsingular. The controllability **Gramian** is also the solution to the following integral equation:*

$$\mathcal{C}_g = \int_0^\infty e^{\tau\mathbf{F}}\mathbf{G}\mathbf{G}^T e^{\tau\mathbf{F}^T} d\tau. \tag{F.26}$$

One physical interpretation of the controllability Gramian is that if the input to the system is white Gaussian noise, then $\mathcal{C}_g$ is the covariance of the state. The controllability Gramian (for an asymptotically stable system) can be computed with the following command in MATLAB: [cg]=gram(F,G).

In conclusion, the four definitions for controllability—pole assignment (Definition I), state reachability (Definition II), mode coupling to the input (Definition III), and controllability Gramian (Definition IV)—are equivalent. The tests for any of these four properties are found in terms of the rank of the controllability or controllability Gramian matrices or the rank of the **matrix pencil** [s**I** - **F** **G**]. If $\mathcal{C}$ is nonsingular, then we can assign the closed-loop poles arbitrarily by state feedback, we can move the state to any point in the state space in a finite time, and every mode is connected to the control input.[2]

F.2 Observability

So far we have discussed only controllability. The concept of observability is parallel to that of controllability, and all of the results we have discussed thus far may be transformed to statements about observability by invoking the property of duality, as discussed in Section 7.7.2. The observability definitions analogous to those for controllability are as follows:

1. *Definition I*: The system (**F**, **H**) is **observable** if, for any nth-order polynomial $\alpha_e(s)$, there exists an estimator gain **L** such that the characteristic equation of the state estimator error is $\alpha_e(s)$.

2. *Definition II*: The system (**F**, **H**) is **observable** if, for any **x(0),** there is a finite time τ such that **x(0)** can be determined (uniquely) from $u(t)$ and $y(t)$ for $0 \le t \le \tau$.

3. *Definition III*: The system (**F**, **H**) is **observable** if every dynamic mode in **F** is connected to the output through **H**.

4. *Definition IV*: The asymptotically stable system (**F**, **H**) is **observable** if the observability Gramian is nonsingular.

As we saw in the discussion for controllability, mathematical tests can be developed for observability. The system is observable if the observability matrix

$$\mathcal{O} = \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \vdots \\ \mathbf{HF}^{n-1} \end{bmatrix} \tag{F.27}$$

is nonsingular. If we take the transpose of $\mathcal{O}$ and let $\mathbf{H}^T = \mathbf{G}$ and $\mathbf{F}^T = \mathbf{F}$, then we find the controllability matrix of (**F**, **G**), another manifestation of duality. The observability matrix $\mathcal{O}$ may

[2] We have shown the latter for diagonal **F** only, but the result is true in general.

be computed using the obsv command in MATLAB: [oo]=obsv(F,H). The system ($\mathbf{F}$, $\mathbf{H}$) is observable if the following **matrix pencil** is full rank for all s:

$$\text{rank} \begin{bmatrix} s\mathbf{I} - \mathbf{F} \\ \mathbf{H} \end{bmatrix} = n. \tag{F.28}$$

The observability Gramian $\mathcal{O}_g$, which is a symmetric matrix, and the solution to the integral equation

$$\mathcal{O}_g = \int_0^\infty e^{\tau \mathbf{F}^T} \mathbf{H}^T \mathbf{H} e^{\tau \mathbf{F}} \, d\tau, \tag{F.29}$$

as well as the Lyapunov equation

$$\mathbf{F}^T \mathcal{O}_g + \mathcal{O}_g \mathbf{F} + \mathbf{H}^T \mathbf{H} = \mathbf{0}, \tag{F.30}$$

can also be computed (for an asymptotically stable system) using the gram command in MATLAB: [og]=gram(F',H'). The observability Gramian has an interpretation as the "information matrix" in the context of estimation.

Appendix G

Ackermann's Formula for Pole Placement

Given the plant and state-variable equation

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u}, \tag{G.1}$$

our objective is to find a state-feedback control law

$$u = -\mathbf{K}\mathbf{x} \tag{G.2}$$

such that the closed-loop characteristic polynomial is

$$\alpha_c(s) = \det(s\mathbf{I} - \mathbf{F} + \mathbf{G}\mathbf{K}). \tag{G.3}$$

First we have to select $\alpha_c(s)$, which determines where the poles are to be shifted; then we have to find $\mathbf{K}$ such that Eq. (G.3) will be satisfied. Our technique is based on transforming the plant equation into control canonical form.

We begin by considering the effect of an arbitrary nonsingular transformation of the state,

$$\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}, \tag{G.4}$$

where $\bar{\mathbf{x}}$ is the new transformed state. The equations of motion in the new coordinates, from Eq. (G.4), are

$$\dot{\mathbf{x}} = \mathbf{T}\dot{\bar{\mathbf{x}}} = \mathbf{F}\mathbf{x} + \mathbf{G}u = \mathbf{F}\mathbf{T}\bar{\mathbf{x}} + \mathbf{G}u, \tag{G.5}$$

$$\dot{\bar{\mathbf{x}}} = \mathbf{T}^{-1}\mathbf{F}\mathbf{T}\bar{\mathbf{x}} + \mathbf{T}^{-1}\mathbf{G}u = \bar{\mathbf{F}}\bar{\mathbf{x}} + \bar{\mathbf{G}}u. \tag{G.6}$$

Now the controllability matrix for the original state,

$$\mathcal{C}_x = [\ \mathbf{G} \quad \mathbf{F}\mathbf{G} \quad \mathbf{F}^2\mathbf{G} \quad \cdots \quad \mathbf{F}^{n-1}\mathbf{G}\ ], \tag{G.7}$$

provides a useful transformation matrix. We can also define the controllability matrix for the transformed state:

$$\mathcal{C}_{\bar{\mathbf{x}}} = [\ \bar{\mathbf{G}} \quad \bar{\mathbf{F}}\bar{\mathbf{G}} \quad \bar{\mathbf{F}}^2\bar{\mathbf{G}} \quad \cdots \quad \bar{\mathbf{F}}^{n-1}\bar{\mathbf{G}}\ ]. \tag{G.8}$$

The two controllability matrices are related by

$$\mathcal{C}_{\bar{\mathbf{x}}} = [\ \bar{\mathbf{T}}^{-1}\mathbf{G} \quad \mathbf{T}^{-1}\mathbf{F}\mathbf{T}\mathbf{T}^{-1}\mathbf{G} \quad \cdots\ ] = \mathbf{T}^{-1}\mathcal{C}_{\mathbf{x}} \tag{G.9}$$

and the transformation matrix

$$\mathbf{T} = \mathcal{C}_x\mathcal{C}_{\bar{\mathbf{x}}}^{-1}. \tag{G.10}$$

From Eqs. (G.9) and (G.10) we can draw some important conclusions. From Eq. (G.9), we see that if $\mathcal{C}_{\mathbf{x}}$ is nonsingular, then for any nonsingular $\mathbf{T}$, $\mathcal{C}_{\bar{\mathbf{x}}}$ is also nonsingular. This means that a similarity transformation on the state does not change the controllability properties of a system. We can look at this in another way. Suppose we would like to find a transformation to take the system $(\mathbf{F}, \mathbf{G})$ into control canonical form. As we shall shortly see, $\mathcal{C}_{\bar{\mathbf{x}}}$ in that case is *always* nonsingular. From Eq. (G.9) we see that a nonsingular $\mathbf{T}$ will always exist if and only if $\mathcal{C}_{\mathbf{x}}$ is nonsingular. We conclude that

Theorem G.1 *We can always transform $(\mathbf{F}, \mathbf{G})$ into control canonical form if and only if the system is controllable.*

Let us take a closer look at control canonical form and treat the third-order case, although the results are true for any nth-order case:

$$\bar{\mathbf{F}} = \mathbf{F}_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{\mathbf{G}} = \mathbf{G}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{G.11}$$

The controllability matrix, by direct computation, is

$$\mathcal{C}_{\bar{\mathbf{x}}} = \mathcal{C}_c = \begin{bmatrix} 1 & -a_1 & a_1^2 - a_2 \\ 0 & 1 & -a_1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{G.12}$$

Because this matrix is upper triangular with ones along the diagonal, it is always invertible. Also note that the last row of $\mathcal{C}_{\bar{\mathbf{x}}}$ is the unit vector with all zeros, except for the last element, which is unity. We shall use this fact in what we do next.

As we pointed out in Section 7.5, the design of a control law for the state $\bar{\mathbf{x}}$ is trivial if the equations of motion happen to be in control canonical form. The characteristic equation is

$$s^3 + a_1 s^2 + a_2 s + a_3 = 0, \tag{G.13}$$

and the characteristic equation for the closed-loop system comes from

$$\mathbf{F}_{cl} = \mathbf{F}_c - \mathbf{G}_c \mathbf{K}_c \tag{G.14}$$

and has the coefficients shown:

$$s^3 + (a_1 + K_{c1})s^2 + (a_2 + K_{c2})s + (a_3 + K_{c3}) = 0. \tag{G.15}$$

To obtain the desired closed-loop pole locations, we must make the coefficients of s in Eq. (G.15) match those in

$$\alpha_c(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3, \tag{G.16}$$

so

$$a_1 + K_{c1} = \alpha_1, \quad a_2 + K_{c2} = \alpha_2, \quad a_3 + K_{c3} = \alpha_3, \tag{G.17}$$

or, in vector form,

$$\mathbf{a} + \mathbf{K}_c = \boldsymbol{\alpha}, \tag{G.18}$$

where $\mathbf{a}$ and $\boldsymbol{\alpha}$ are row vectors containing the coefficients of the characteristic polynomials of the open-loop and closed-loop systems, respectively.

We now need to find a relationship between these polynomial coefficients and the matrix $\mathbf{F}$. The requirement is achieved by the Cayley–Hamilton theorem, which states that a matrix satisfies its own characteristic polynomial. For $\mathbf{F_c}$ this means that

$$\mathbf{F}_c^n + a_1 \mathbf{F}_c^{n-1} + a_2 \mathbf{F}_c^{n-2} + \cdots + a_n \mathbf{I} = \mathbf{0}. \tag{G.19}$$

Now suppose we form the polynomial $\alpha_c(\mathbf{F})$, which is the *closed-loop* characteristic polynomial with the matrix $\mathbf{F}$ substituted for the complex variable s:

$$\alpha_c(\mathbf{F}_c) = \mathbf{F}_c^n + \alpha_1\mathbf{F}_c^{n-1} + \alpha_2\mathbf{F}_c^{n-2} + \cdots + \alpha_n\mathbf{I}. \tag{G.20}$$

If we solve Eq. (G.19) for $\mathbf{F}_c^n$ and substitute into Eq. (G.20), we find that

$$\alpha_c(\mathbf{F}_c) = (-a_1 + \alpha_1)\mathbf{F}_c^{n-1} + (-a_2 + \alpha_2)\mathbf{F}_c^{n-2} + \cdots + (-\alpha_n + \alpha_n)\mathbf{I}. \tag{G.21}$$

But, because $\mathbf{F}_c$ has such a special structure, we observe that if we multiply it by the transpose of the nth unit vector, $\mathbf{e}_n^T = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$, we get

$$\mathbf{e}_n^T\mathbf{F}_c = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 \end{bmatrix} = \mathbf{e}_{n-1}^T, \tag{G.22}$$

as we can see from Eq. (G.11). If we multiply this vector again by $\mathbf{F}_c$, getting

$$\begin{aligned}(\mathbf{e}_n^T\mathbf{F}_c)\mathbf{F}_c &= \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 \end{bmatrix}\mathbf{F}_c \\ &= \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & 0 \end{bmatrix} = \mathbf{e}_{n-2}^T,\end{aligned} \tag{G.23}$$

and continue the process, successive unit vectors are generated until

$$\mathbf{e}_n^T\mathbf{F}_c^{n-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} = \mathbf{e}_1^T. \tag{G.24}$$

Therefore, if we multiply Eq. (G.21) by $\mathbf{e}_n^T$, we find that

$$\begin{aligned}\mathbf{e}_n^T\alpha_c(\mathbf{F}_c) &= (-a_1 + \alpha_1)\mathbf{e}_1^T + (-a_2 + \alpha_2)\mathbf{e}_2^T + \cdots + (-a_n + \alpha_n)\mathbf{e}_n^T \\ &= \begin{bmatrix} K_{c1} & K_{c2} & \cdots & K_{cn} \end{bmatrix} = \mathbf{K}_c,\end{aligned} \tag{G.25}$$

where we use Eq. (G.18), which relates $\mathbf{K}_c$ to $\mathbf{a}$ and $\boldsymbol{\alpha}$.

We now have a compact expression for the gains of the system in control canonical form as represented in Eq. (G.25). However, we still need the expression for $\mathbf{K}$, the gain on the original state. If $u = -\mathbf{K}_c\bar{\mathbf{x}}$, then $u = -\mathbf{K}_c\mathbf{T}^{-1}\mathbf{x}$, so that

$$\begin{aligned}\mathbf{K} = \mathbf{K}_c\mathbf{T}^{-1} &= \mathbf{e}_n^T\alpha_c(\mathbf{F}_c)\mathbf{T}^{-1} \\ &= \mathbf{e}_n^T\alpha_c(\mathbf{T}^{-1}\mathbf{F}\mathbf{T})\mathbf{T}^{-1} \\ &= \mathbf{e}_n^T\mathbf{T}^{-1}\alpha_c(\mathbf{F}).\end{aligned} \tag{G.26}$$

In the last step of Eq. (G.26) we used the fact that $(\mathbf{T}^{-1}\mathbf{F}\mathbf{T})^k = \mathbf{T}^{-1}\mathbf{F}^k\mathbf{T}$ and that α_c is a polynomial, that is, a sum of the powers of $\mathbf{F}_c$. From Eq. (G.9) we see that

$$\mathbf{T}^{-1} = \mathcal{C}_c\mathcal{C}_{\mathbf{x}}^{-1}. \tag{G.27}$$

With this substitution, Eq. (G.26) becomes

$$\mathbf{K} = \mathbf{e}_n^T\mathcal{C}_c\mathcal{C}_{\mathbf{x}}^{-1}\alpha_c(\mathbf{F}). \tag{G.28}$$

Now, we use the observation made earlier for Eq. (G.12) that the last row of $\mathcal{C}_c$, which is $\mathbf{e}_n^T\mathcal{C}_c$, is again $\mathbf{e}_n^T$. We finally obtain Ackermann's formula:

Ackermann's formula

$$\mathbf{K} = \mathbf{e}_n^T\mathcal{C}_{\mathbf{x}}^{-1}\alpha_c(\mathbf{F}). \tag{G.29}$$

We note again that forming the explicit inverse of $\mathcal{C}_{\mathbf{x}}$ is not advisable for numerical accuracy. Thus we need to solve $\mathbf{b}^T$ such that

$$\mathbf{e}_n^T\mathcal{C}_{\mathbf{x}}^{-1} = \mathbf{b}^T. \tag{G.30}$$

We solve the linear set of equations

$$\mathbf{b}^T\mathcal{C}_x = \mathbf{e}_n^T \tag{G.31}$$

and then compute

$$\mathbf{K} = \mathbf{b}^T\alpha_c(\mathbf{F}). \tag{G.32}$$

Ackermann's formula, Eq. (G.29), even though elegant, is not recommended for systems with a large number of state variables. Even if it is used, Eqs. (G.31) and (G.32) are recommended for better numerical accuracy.