Appendix W 3

Block Diagram Reduction

W.3 △Mason's Rule and the Signal-Flow Graph

Signal-flow graph

A compact alternative notation to the block diagram is given by the **signal-flow graph** introduced by S. J. Mason (1953, 1956). As with the block diagram, the signal-flow graph offers a visual tool for representing the causal relationships between the components of the system. The method consists of characterizing the system by a network of directed branches and associated gains (transfer functions) connected at nodes. Several block diagrams and their corresponding signal-flow graphs are shown in Fig. W.1. The two ways of depicting a system are equivalent, and you can use either diagram to apply Mason's rule (to be defined shortly).

In a signal-flow graph the internal signals in the diagram, such as the common input to several blocks or the output of a summing junction, are called **nodes**. The system input point and the system output point are also nodes; the input node has outgoing branches only, and the output node has incoming branches only. Mason defined a **path** through a block diagram as a sequence of connected blocks, the route passing from one node to another *in the direction of signal flow of the blocks* without including any block more than once. A **forward path** is a path from the input to output such that no node is included more than once. If the nodes are numbered in a convenient order, then a forward path can be identified by the numbers that are included. Any closed path that returns to its starting node without passing through any node more than once is a **loop**, and a path that leads from a given variable back to the same variable is a **loop path**. The **path gain** is the product of component gains (transfer functions) making up the path. Similarly, the **loop gain** is the path gain associated with a loop—that is, the product of gains in a loop. If two paths have a common component, they are said to touch. Notice particularly in this connection that the input and the output of a summing junction are not the same and that the summing junction is a one-way device from its inputs to its output.

Mason's rule relates the graph to the algebra of the simultaneous equations it represents.[1] Consider Fig. W.1(c), where the signal at each node has been given a name and the gains are marked. Then the block diagram (or the signal-flow graph) represents the following system of equations:

$$\begin{aligned} X_1(s) &= X_3(s) + U(s), \\ X_2(s) &= G_1(s)X_1(s) + G_2(s)X_2(s) + G_4(s)X_3(s), \\ Y(s) &= 1X_3(s). \end{aligned}$$

Mason's rule

Mason's rule states that the input–output transfer function associated with a signal-flow graph is given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{\Delta}\sum_i G_i \Delta_i,$$

[1] The derivation is based on Cramer's rule for solving linear equations by determinants and is described in Mason's papers.

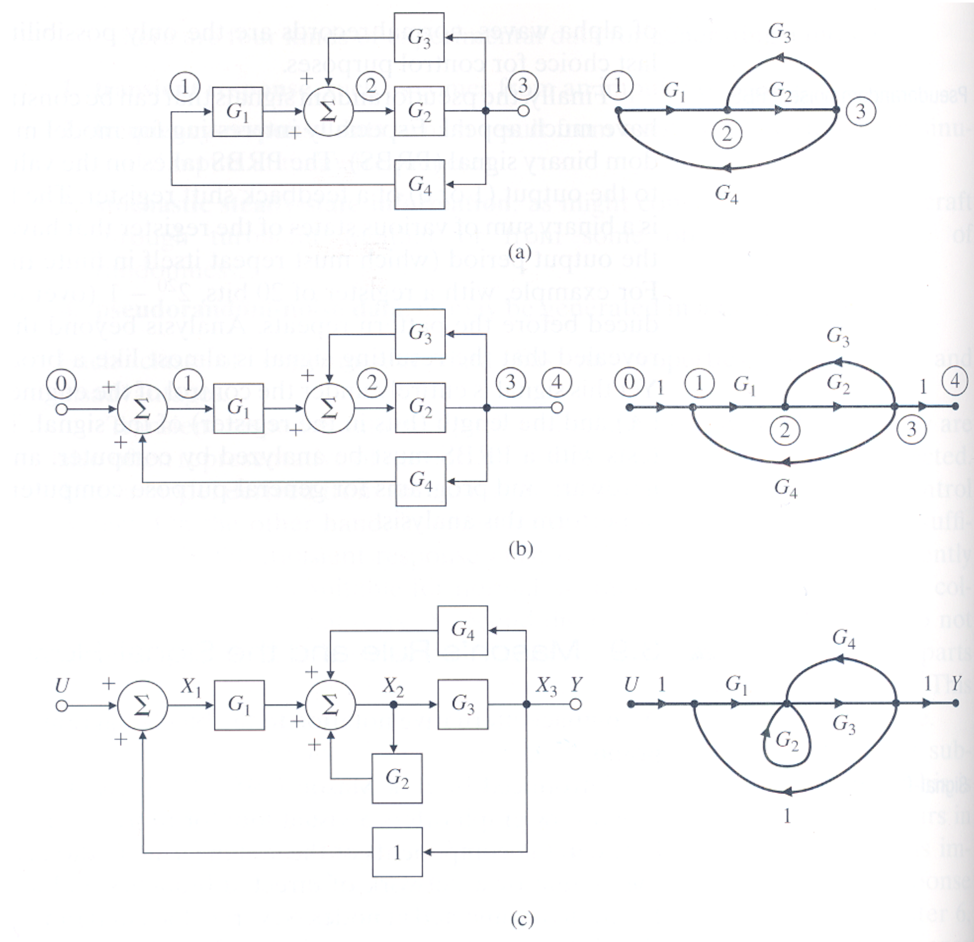

Figure W.1: Block diagrams and corresponding signal flow graphs where

$$\begin{aligned}
G_i &= \text{path gain of the } i\text{th forward path,}\\
\Delta &= \text{the system determinant}\\
&= 1-\sum\text{(all individual loop gains)} + \sum\text{(gain products of all possible two loops that do not touch)} - \sum\text{(gain products of all possible three loops that do not touch)} + \dots,\\
\Delta_i &= i\text{th forward path determinant}\\
&= \text{value of } \Delta \text{ for that part of the block diagram that does } \textit{not} \text{ touch the } i\text{th forward path.}
\end{aligned}$$

We will now illustrate the use of Mason's rule by several examples.

Example W.1 *Mason's Rule in a Simple System Find the transfer function for the block diagram in Fig. W.2.*

SOLUTION From the block diagram shown in Fig. W.2 we have

Forward Path	*Path Gain*
1236	$G_1 = 1\left(\frac{1}{s}\right)(b_1)(1)$
12346	$G_2 = 1\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)(b_2)(1)$
123456	$G_3 = 1\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)(b_3)(1)$
	Loop Path Gain
232	$l_1 = -a_1/s$
2342	$l_2 = -a_2/s^2$
23452	$l_3 = -a_3/s^3$

and the determinants are

$$\begin{aligned}
\Delta &= 1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3}\right) + 0\\
\Delta_1 &= 1 - 0\\
\Delta_2 &= 1 - 0\\
\Delta_3 &= 1 - 0.
\end{aligned}$$

Applying Mason's rule, we find the transfer function to be

$$\begin{aligned}
G(s) &= \frac{Y(s)}{U(s)} = \frac{(b_1/s) + (b_2/s^2) + (b_3/s^3)}{1 + (a_1/s) + (a_2/s^2) + (a_3/s^3)}\\
&= \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.
\end{aligned}$$

Mason's rule is particularly useful for more complex systems where there are several loops, some of which do not sum into the same point.

Example W.2 *Mason's Rule in a Complex System Find the transfer function for the system shown in Fig. W.3.*

SOLUTION From the block diagram, we find that

Forward Path	*Path Gain*
12456	$G_1 = H_1 H_2 H_3$
1236	$G_2 = H_4$
	Loop Path Gain
242	$l_1 = H_1 H_5$ *(does not touch* l_3*)*

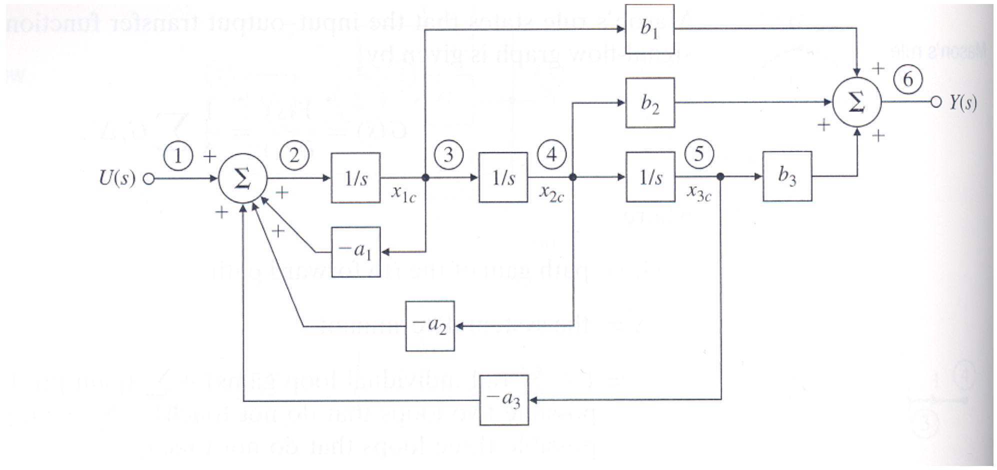

Figure W.2: Block diagram for Example W.1

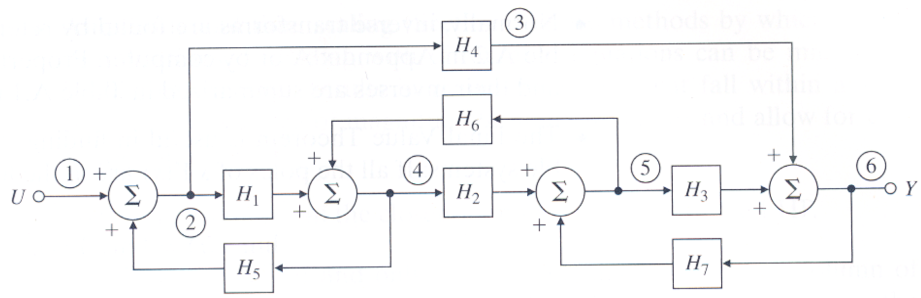

Figure W.3: Block diagram for Example W.2

454 $\quad l_2 = H_2H_6$

565 $\quad l_3 = H_3H_7$ (*does not touch* l_1)

236542 $\quad l_4 = H_4H_7H_6H_5$

and the determinants are

$$\begin{aligned}\Delta &= 1-(H_1H_5+H_2H_6+H_3H_7+H_4H_7H_6H_5)+(H_1H_5H_3H_7)\\ \Delta_1 &= 1-0\\ \Delta_2 &= 1-H_2H_6.\end{aligned}$$

Therefore,

$$G(s)=\frac{Y(s)}{U(s)}=\frac{H_1H_2H_3+H_4-H_4H_2H_6}{1-H_1H_5-H_2H_6-H_3H_7-H_4H_7H_6H_5+H_1H_5H_3H_7}.$$

Mason's rule is useful for solving relatively complicated block diagrams by hand. It yields the solution in the sense that it provides an explicit input–output relationship for the system represented by the diagram. The advantage compared with path-by-path block-diagram reduction is that it is systematic and algorithmic rather than problem dependent. MATLAB and other control systems computer-aided software allow you to specify a system in terms of individual blocks in an overall

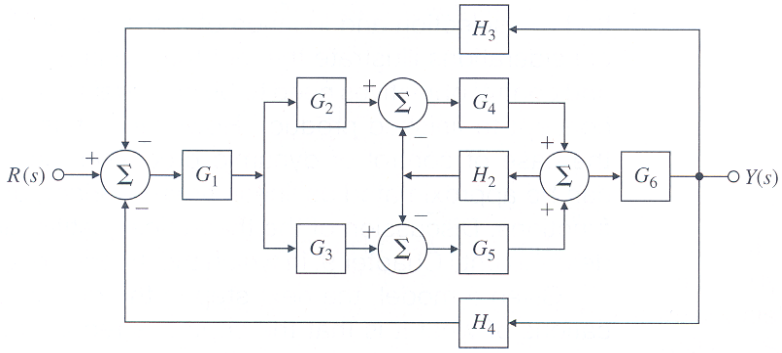

Figure W.4: Block diagram for Problem 2

system, and the software algorithms perform the required block-diagram reduction; therefore, Mason's rule is less important today than in the past. However, there are some derivations that rely on the concepts embodied by the rule, so it still has a role in the control designer's toolbox.

Problems: Mason's Rule and the Signal-Flow Graph

1. △ Find the transfer functions for the block diagrams in Fig. 3.53, using Mason's rule.
2. △ Use block-diagram algebra or Mason's rule to determine the transfer function between *R(s)* and *Y(s)* in Fig. W.4.