Appendix W 8

Digital Control

W.8 State-Space Design Methods

We have seen in previous chapters that a linear, constant-coefficient continuous system can be represented by a set of first-order matrix differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u, \tag{W.1}$$

where u is the control input to the system. The output equation can be expressed as

$$y = \mathbf{H}\mathbf{x} + Ju. \tag{W.2}$$

The solution to these equations (see Franklin *et al.,* 1998) is

$$\mathbf{x}(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)}\mathbf{G}u(\tau)\,d\tau. \tag{W.3}$$

It is possible to use Eq. (W.3) to obtain a discrete state-space representation of the system. Because the solution over one sample period results in a difference equation, we can alter the notation a bit (letting $t = kT + T$ and $t_0 = kT$) to arrive at a particularly useful version of Eq. (W.3):

$$\mathbf{x}(kT+T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)\,d\tau. \tag{W.4}$$

This result is not dependent on the type of hold, because u is specified in terms of its continuous time history $u(\tau)$ over the sample interval. To find the discrete model of a continuous system where the input $u(t)$ is the output of a ZOH, we let $u(\tau)$ be a constant throughout the sample interval—that is,

$$u(\tau) = u(kT), kT \le \tau < kT + T.$$

To facilitate the solution of Eq. (W.4) for a ZOH, we let

$$\eta = kT + T - \tau,$$

which converts Eq. (W.4) to

$$\mathbf{x}(kT+T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \left(\int_0^T e^{\mathbf{F}\eta}\,d\eta\right)\mathbf{G}u(kT).$$

If we let

$$\boldsymbol{\Phi} = e^{\mathbf{F}T}$$

and

$$\boldsymbol{\Gamma} = \left(\int_0^T e^{\mathbf{F}\eta}\,d\eta\right)\mathbf{G}, \tag{W.5}$$

Eqs. (W.4) and (W.2) reduce to difference equations in standard form:

Difference equations in standard form

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k), \tag{W.6}$$
$$y(k) = \mathbf{H}\mathbf{x}(k) + Ju(k). \tag{W.7}$$

Here $\mathbf{x}(k+1)$ is a shorthand notation for $\mathbf{x}(kT+T)$, $\mathbf{x}(k)$ for $\mathbf{x}(kT)$, and $u(k)$ for $u(kT)$. The series expansion

$$\mathbf{\Phi} = e^{\mathbf{F}T} = \mathbf{I} + \mathbf{F}T + \frac{\mathbf{F}^2T^2}{2!} + \frac{\mathbf{F}^3T^3}{3!} + \cdots$$

can also be written

$$\mathbf{\Phi} = \mathbf{I} + \mathbf{F}T\mathbf{\Psi}, \tag{W.8}$$

where

$$\mathbf{\Psi} = \mathbf{I} + \frac{\mathbf{F}T}{2!} + \frac{\mathbf{F}^2T^2}{3!} + \cdots .$$

The $\mathbf{\Gamma}$ integral in Eq. (W.5) can be evaluated term by term to give

$$\begin{aligned}\mathbf{\Gamma} &= \sum_{k=0}^{\infty} \frac{\mathbf{F}^kT^{k+1}}{(k+1)!}\mathbf{G} \\ &= \sum_{k=0}^{\infty} \frac{\mathbf{F}^kT^{k}}{(k+1)!}T\mathbf{G} \\ &= \mathbf{\Psi}T\mathbf{G}.\end{aligned} \tag{W.9}$$

We evaluate $\mathbf{\Psi}$ by a series in the form

$$\mathbf{\Psi} \cong \mathbf{I} + \frac{\mathbf{F}T}{2}\left\{\mathbf{I} + \frac{\mathbf{F}T}{3}\left[\mathbf{I} + \cdots \frac{\mathbf{F}T}{N-1}\left(\mathbf{I} + \frac{\mathbf{F}T}{N}\right)\right]\right\},$$

MATLAB c2d

which has better numerical properties than the direct series. We then find $\mathbf{\Gamma}$ from Eq. (W.9) and $\mathbf{\Phi}$ from Eq. (W.8). For a discussion of various methods of numerical determination of $\mathbf{\Phi}$ and $\mathbf{\Gamma}$, see Franklin et al. (1998) and Moler and van Loan (1978, 2003). The evaluation of the $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ matrices in practice is carried out by the `c2d` function in MATLAB.

To compare this method of representing the plant with the discrete transfer function, we can take the z-transform of Eqs. (W.6) and (W.7) with $J = 0$ to obtain

$$(z\mathbf{I} - \mathbf{\Phi})\mathbf{X}(z) = \mathbf{\Gamma}U(z), \tag{W.10}$$
$$Y(z) = \mathbf{H}\mathbf{X}(z). \tag{W.11}$$

Therefore,

$$\frac{Y(z)}{U(z)} = G(z) = \mathbf{H}(z\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma}. \tag{W.12}$$

Example W.1 *Discrete State-Space Representation of $1/s^2$ Plant Use the relation in this section to verify that the discrete model of the $1/s^2$ plant preceded by a ZOH is that given in the solution to Example 8.4.*

SOLUTION The $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ matrices can be calculated using Eqs. (W.8) and (W.9). Example 7.1 (with $I = 1$) showed that the values for $\boldsymbol{F}$ and $\boldsymbol{G}$ are

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Because $\mathbf{F}^2 = \mathbf{0}$ *in this case, we have*

$$\begin{aligned}
\mathbf{\Phi} &= \mathbf{I} + \mathbf{F}T + \frac{{}^2T^2}{2!} + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} T = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \\
\mathbf{\Gamma} &= \left(\mathbf{I} + \mathbf{F}\frac{T}{2!} \right) T\mathbf{G} \\
&= \left(\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{T^2}{2} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}.
\end{aligned}$$

Hence, using Eq. (W.12), we obtain

$$\begin{aligned}
G(z) = \frac{Y(z)}{U(z)} &= [1 \quad 0] \left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \\
&= \frac{T^2}{2} \left[\frac{z+1}{(z-1)^2} \right]. \qquad \text{(W.13)}
\end{aligned}$$

This is the same result we obtained using Eq. (8.33) and the z*-transform tables in Example 8.4.*

Note that to compute Y/U*, we find that the denominator of Eq. (W.13) is* $\det(z\mathbf{I} - \mathbf{\Phi})$*, which was created by the matrix inverse in Eq. (W.12). This determinant is the characteristic polynomial of the transfer function, and the zeros of the determinant are the poles of the plant. We have two poles at* $z = 1$ *in this case, corresponding to two integrations in this plant's equations of motion.*

We can further explore the question of poles and zeros and the state-space description by considering again the transform formulas (Eqs. (W.10) and (W.11)). One way to interpret transfer-function poles from the perspective of the corresponding difference equation is that a pole is a value of z such that the equation has a nontrivial solution when the forcing input is zero. From Eq. (W.10), this interpretation implies that the linear equations

$$(z\mathbf{I} - \mathbf{\Phi})\mathbf{X}(z) = \mathbf{0}$$

have a nontrivial solution. From matrix algebra the well-known requirement for a nontrivial solution is that $\det(z\mathbf{I} - \mathbf{\Phi}) = 0$. Using the system in Example W.1, we get

$$\begin{aligned}
\det(z\mathbf{I} - \mathbf{\Phi}) &= \det \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \right) \\
&= \det \begin{bmatrix} z-1 & -T \\ 0 & z-1 \end{bmatrix} \\
&= (z-1)^2 = 0,
\end{aligned}$$

which is the characteristic equation, as we have seen. In MATLAB, the poles of the system are found by P = eig(Phi).

Along the same line of reasoning, a system zero is a value of z such that the system output is zero even with a nonzero state-and-input combination. Thus, if we are able to find a nontrivial solution for $\mathbf{X}(z_0)$ and $U(z_0)$ such that $Y(z_0)$ is identically zero, then z_0 is a zero of the system. In combining Eqs. (W.10) and (W.11), we must satisfy the requirement that

$$\begin{bmatrix} z\mathbf{I} - \mathbf{\Phi} & -\mathbf{\Gamma} \\ \mathbf{H} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}(z) \\ U(z) \end{bmatrix} = \mathbf{0}.$$

Once more the condition for the existence of nontrivial solutions is that the determinant of the square coefficient system matrix be zero. For Example W.1, the calculation is

$$\begin{aligned}\det\begin{bmatrix} z-1 & -T & -T^2/2 \\ 0 & z-1 & -T \\ 1 & 0 & 0 \end{bmatrix} &= \det\begin{bmatrix} -T & -T^2/2 \\ z-1 & -T \end{bmatrix} \\ &= T^2 + \frac{T^2}{2}(z-1) \\ &= \frac{T^2}{2}z + \frac{T^2}{2} \\ &= \frac{T^2}{2}(z+1).\end{aligned}$$

Thus we have a single zero at $z = -1$, as we have seen from the transfer function. In MATLAB, the zeros are found by `Z=tzero(Phi,Gam,H,J)`.

Much of the algebra for discrete state-space control design is the same as for the continuous time case discussed in Chapter 7. The poles of a discrete system can be moved to desirable locations by linear state-variable feedback

$$u = -\mathbf{K}\mathbf{x}$$

such that

$$\det(z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma}\mathbf{K}) = \alpha_c(z), \tag{W.14}$$

provided that the system is controllable. The system is controllable if the controllability matrix

$$C = [\ \mathbf{\Gamma} \quad \mathbf{\Phi}\mathbf{\Gamma} \quad \mathbf{\Phi}^2\mathbf{\Gamma} \quad \ldots \quad \mathbf{\Phi}^{n-1}\mathbf{\Gamma} \quad]$$

is full rank.

A discrete full-order estimator has the form

$$\bar{\mathbf{x}}(k+1) = \mathbf{\Phi}\bar{\mathbf{x}}(k) + \mathbf{\Gamma}u(k) + \mathbf{L}[y(k) - \mathbf{H}\bar{\mathbf{x}}(k)],$$

where $\bar{\mathbf{x}}$ is the state estimate. The error equation,

$$\tilde{\mathbf{x}}(k+1) = (\mathbf{\Phi} - \mathbf{L}\mathbf{H})\tilde{\mathbf{x}}(k),$$

can be given arbitrary dynamics $\alpha_e(z)$, provided that the system is observable, which requires that the observability matrix

$$\mathcal{O} = \begin{bmatrix} \mathbf{H} \\ \mathbf{H}\mathbf{\Phi} \\ \mathbf{H}\mathbf{\Phi}^2 \\ \vdots \\ \mathbf{H}\mathbf{\Phi}^{n-1} \end{bmatrix}$$

be full rank.

As was true for the continuous-time case, if the open-loop transfer function is

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b(z)}{a(z)},$$

then a state-space compensator can be designed such that

$$\frac{Y(z)}{R(z)} = \frac{K_s\gamma(z)b(z)}{\alpha_c(z)\alpha_e(z)},$$

where r is the reference input. The polynomials $\alpha_c(z)$ and $\alpha_e(z)$ are selected by the designer using exactly the same methods discussed in Chapter 7 for continuous systems. $\alpha_c(z)$ results in a control gain $\mathbf{K}$ such that $\det(z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma K}) = \alpha_c(z)$, and $\alpha_e(z)$ results in an estimator gain $\mathbf{L}$ such that $\det(\mathbf{zI} - \mathbf{\Phi} + \mathbf{LH}) = \alpha_e(z)$. If the estimator is structured according to Fig. 7.48(a), the system zeros $\gamma(z)$ will be identical to the estimator poles $\alpha_e(z)$, thus removing the estimator response from the closed-loop system response. However, if desired, we can arbitrarily select the polynomial $\gamma(z)$ by providing suitable feed-forward from the reference input. Refer to Franklin et al. (1998) for details.

Example W.2 *State-Space Design of a Digital Controller Design a digital controller for a* $1/s^2$ *plant to meet the specifications given in Example 8.2. Use state-space design methods, including the use of an estimator, and structure the reference input in two ways: (a) Use the error command shown in Fig. 7.48(b), and (b) use the state command shown in Fig.7.15 and Fig. 7.48(a).*

SOLUTION We find the state-space model of the $1/s^2$ *plant preceded by a ZOH using the MATLAB statements*

```
sysSSc = ss([0 1;0 0], [0; 1], [1 0], 0];
T = 1;
sysSSd = c2d(sysSSc, T);
[Phi,Gam,H] = ssdata(sysSSd);
```

Using discrete analysis for Example 8.4, we find that the desired z-plane roots are at $z = 0.78 \pm 0.18j$. *Solving the discrete pole-placement problem involves placing the eigenvalues of* $\mathbf{\Phi} - \mathbf{\Gamma K}$, *as indicated by Eq. (W.14). Likewise, the solution of the continuous pole-placement problem involves placing the eigenvalues of* $\mathbf{F} - \mathbf{GK}$, *as indicated by Eq. (7.72). Because these two tasks are identical, we use the same function in MATLAB for the continuous and discrete cases. Therefore, the control feedback matrix* $\mathbf{K}$ *is found by*

```
pc = [0.78 + 0.18*j; 0.78 - 0.18*j];
K = acker(Phi,Gam,pc);
```

which yields

$$\mathbf{K} = [\ 0.0808 \quad 0.3996\ ].$$

To ensure that the estimator roots are substantially faster than the control roots (so that the estimator roots will have little effect on the output), we choose them to be at $z = 0.2 \pm 0.2j$. *Therefore, the estimator feedback matrix* $\mathbf{L}$ *is found by*

```
pe = [0.2 + 0.2*j; 0.2 - 0.2*j];
L = acker(Phi', H', pe');
```

which yields

$$\mathbf{L} = \begin{bmatrix} 1.6 \\ 0.68 \end{bmatrix}.$$

The equations of the compensation for $r = 0$ *(regulation to* $\mathbf{x}^T = [0 \quad 0]$*) are then*

$$\bar{\mathbf{x}}(k+1) = \mathbf{\Phi}\bar{\mathbf{x}}(k) + \mathbf{\Gamma}u(k) + \mathbf{L}[y(k) - \mathbf{H}\bar{\mathbf{x}}(k)], \tag{W.15}$$

$$u(k) = -\mathbf{K}\bar{\mathbf{x}}(k). \tag{W.16}$$

1. *For the error command structure where the compensator is placed in the feed-forward path, as shown in Fig. 7.48(b) in the book,* $y(k)$ *from Eq. (W.15) is replaced*

   with $y(k) - r$, *so the state description of the plant plus the estimator (a fourth-order system whose state vector is* $[\mathbf{x}\,\bar{\mathbf{x}}]^T$*) is*

   ```
   A = [Phi    - Gam*K; L*H    Phi - Gam*K - L*H];
   B = [0; 0; -L];
   C = [1 0 0 0];
   D = 0;
   step(A,B,C,D).
   ```

   The resulting step response in Fig. W.1 shows a response similar to that of the step responses in Fig. 8.19 in the book.

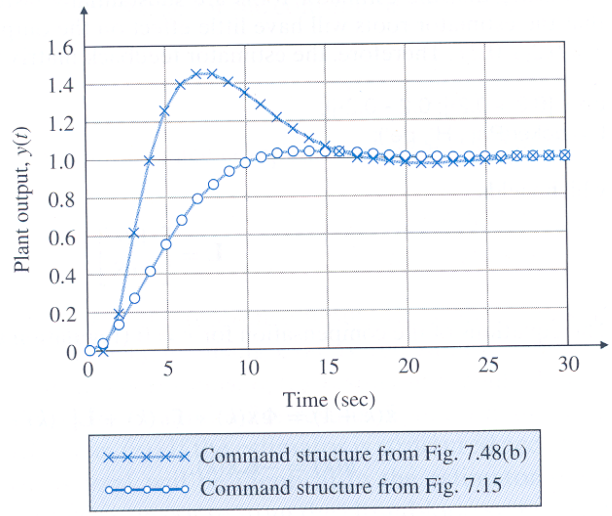

Figure W.1: Step response of Example W.2

2. *For the state command structure described in Section 7.9 in the book, we wish to command the position element of the state vector so that*

$$\mathbf{N_x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and the $1/s^2$ *plant requires no steady control input for a constant output* y*. Therefore* $N_u = 0$*. To analyze a system with this command structure, we need to modify matrix* B *from the preceding MATLAB statement to properly introduce the reference input* r *according to Fig.7.15. The MATLAB statement*

```
B = [Gam*K*Nx; Gam*K*Nx];
```

channels r *into both the plant and estimator equally, thus not exciting the estimator dynamics. The resulting step response in Fig. W.1 shows a substantial reduction in the overshoot with this structure. In fact, the overshoot is now about 5%, which is expected for a second-order system with* $\zeta \cong 0.7$*. The previous designs all had considerably greater overshoot, because of the effect of the extra zero and pole.*

W.8.1 Summary of State-Space Design

- The continuous state-space form of a differential equation,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}u, \\ y &= \mathbf{H}\mathbf{x} + Ju, \end{aligned}$$

has a discrete counterpart in the difference equations

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k), \\ y(k) &= \mathbf{H}x(k) + Ju(k), \end{aligned}$$

where

$$\begin{aligned} \mathbf{\Phi} &= e^{\mathbf{F}T} \\ \mathbf{\Gamma} &= \left(\int_0^T e^{\mathbf{F}\eta}\, d\eta \right) \mathbf{G}. \end{aligned}$$

These matrices can be computed in MATLAB by `[Phi, Gam] = c2d(F,G,H,J)` and used in state-space **discrete design** methods.

- The pole placement and estimation ideas are identical in the continuous and discrete domains.

W.8.2 Problems

1. In Problem 8.11 in the book we dealt with an experiment in magnetic levitation described by Eq. (8.62) which reduces to
$$\ddot{x} = 1000x + 20i.$$
Let the sampling time be 0.01 sec.
   (a) Use pole placement to design a controller for the magnetic levitator so that the closed-loop system meets the following specifications: settling time, $t_s \leq 0.25$ sec, and overshoot to an initial offset in x that is less than 20%.
   (b) Plot the step response of $x, \tilde{x}$, and i to an initial displacement in x.
   (c) Plot the root locus for changes in the plant gain, and mark the pole locations of your design.
   (d) Introduce a command reference input r (as discussed in Section 7.9) that does not excite the estimate of x. Measure or compute the frequency response from r to the system error $r - x$ and give the highest frequency for which the error amplitude is less than 20% of the command amplitude.

2. *Servomechanism for Antenna Elevation Control*: Suppose it is desired to control the elevation of an antenna designed to track a satellite. A photo of such a system is shown in Fig. W.2, and a schematic diagram is depicted in Fig. W.3. The antenna and drive parts have a moment of inertia J and damping B, arising to some extent from bearing and aerodynamic friction, but mostly from the back emf of the DC drive motor. The equation of motion is
$$J\ddot{\theta} + B\dot{\theta} = T_c + T_d,$$
where
$$T_c = \text{net torque from the drive motor,} \quad (W.17)$$
$$T_d = \text{disturbance torque due to wind.} \quad (W.18)$$

Figure W.2: Satellite-tracking antenna *(Courtesy Space Systems/Loral)*

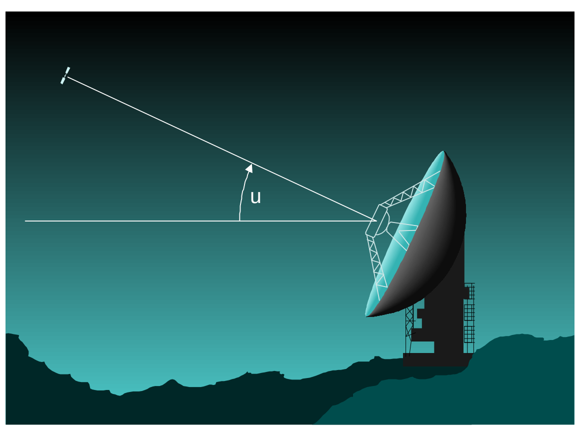

Figure W.3: Schematic diagram of satellite-tracking antenna