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# The Generalized Nyquist Criterion and Robustness Margins with Applications

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*Abstract*— **Multivariable Nyquist eigenloci provide a much richer family of curves as compared to the SISO case. The eigeloci may be computed symbolically in many simple cases. Inspection of the generalized Nyquist eigenloci plot allows the determination of the exact values of the multivariable gain margin (GM), phase margin (PM), and the complex margin (CM). Hence there is no need to compute estimates of GM and PM from the singular values of the sensitivity and the complementary sensitivity functions as they are usually very conservative. Furthermore, the complex margin allows the computation of the minimum additive equal diagonal complex perturbation, and the associated multiplicative complex perturbation that drives the system unstable. All the computations mirror the SISO case. The ideas are applied to semiconductor wafer manufacturing process control and aerospace examples.**

## I. Introduction

The Nyquist criterion is an important stability test with applications to systems, circuits, and networks [1]. It is also the foundation of robust control theory. The beauty of the Nyquist stability criterion lies in the fact that it is a rather simple graphical test. In this paper we explore the use of the generalized Nyquist criterion to study the shapes of multivariable Nyquist loci and compute stability margins. The eigenloci can be computed symbolically for rather simple cases. The multivariable Nyquist eigenloci provide a much richer family of curves as compared to the SISO case [2]. While the shapes include classical plane curves such as the limaçon, cardioid, Cayley's sextic, etc., they seem to have much more complicated shapes not seen in the SISO case. The technique of plotting the eigenloci of the open-loop transfer function matrix allows determination of the stability of multivariable systems for a range of gains by inspection [3]. There are three approaches to the determination of robust stability: (1) $\mu$ analysis [4], (2) excess stability margin $k_m$ [5], and (3) the use of the generalized Nyquist stability criterion. Inspection of the multivariable Nyquist eigenloci plot allows determination of the exact values of the multivariable gain margin (GM) and phase margin (PM) and the complex margin (CM). All the computations mirror the SISO case. This approach makes the computation of GM and PM from singular value plots of the sensitivity and complementary sensitivity functions unnecessary as they are generally very conservative. Furthermore, the complex margin allows computation of the exact minimum additive equal diagonal complex perturbation and the associated multiplicative perturbation that drives the system unstable.

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The closest results to ours appear in [6]. It is true that the generalized stability criterion only determines stability with respect to a single gain in all of the input channels. As such one could argue that this does not characterize the robust stability property for arbitrary perturbations. Nevertheless, we believe that the measures are very helpful in multivariable control system design especially when used in conjunction with more sophisticated robustness analysis approaches such as $\mu$, $k_m$. For example, in many cases in practice the designer is concerned about robustness with respect to actuator uncertainties such as in rapid thermal processing (RTP) [8]. In this application the lamp actuators are identical with the same nonlinearities. These are not "unstructured" uncertainties. Th same gain in all channels is not a significant restriction necessarily as uneven distribution of the gain can also be accommodated by scaling the loop gain $L(s)$ (pre-multiplying by a scaling matrix) and recomputing the robustness margins. Several illustrative examples are provided.

The organization of this paper is as follows. In Section 2 we review the generalized Nyquist criterion. In Section 3 we define the multivariable GM, PM, CM and extend some SISO results to MIMO systems. Examples appear in Section 4. Concluding remarks are in Section 5.

## II. The Generalized Nyquist Stability Criterion

Consider the feedback system shown in Figure 1 where $L(s)$ is a proper square rational $m \times m$ matrix and let $K > 0$. Assume there are no hidden unstable modes of the system. Note that this structure means we have the same gain $K$ in each loop. Let the $m$ eigenvalues of $KL(s)$ be denoted by $\lambda_1, \lambda_2, \ldots, \lambda_m$, which are the solutions to

$$\det\,[\lambda_i(s)I - KL(s)] = 0, \quad i = 1, 2, \ldots, m, \tag{1}$$

and result in

$$\det\,[\lambda I - KL(s)] = \lambda^m + g_1(s)\lambda^{m-1} + \ldots + g_m(s), \tag{2}$$

where the $g_i(s)$ are proper rational functions of $s$. It is known [2] that the *unordered* eigenvalues of a matrix are a continuous function since the eigenvalues of a matrix are a continuous function of its elements.

Theorem 1 [3]: If $KL(s)$ has $P$ unstable poles, then the closed-loop system with return ratio $-KL(s)$ is stable if and only if the eigenloci of $KL(s)$, taken together, encircle the $-1$ point $P$ times in counter-clockwise direction, assuming that there are no hidden modes (unstable pole-zero cancellations). ■

The eigenloci are the loci of the eigenvalues of the loop gain transfer matrix parameterized as a function of frequency.

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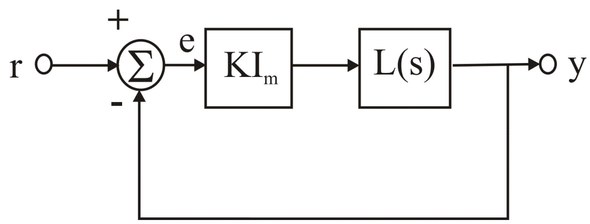


Fig. 1. MIMO unity feedback system.

Theorem 1 suggest that instead of plotting the Nyquist eigenloci of $det(I + KL(s))$, one can plot the eigenloci of $\lambda_i(L(s)), i = 1, 2, ..., m,$ and count the number of the encirclements of the $-1/K$ point as illustrated by the examples in this paper.

## III. Robustness Metrics: Multivariable Gain, Phase, and Complex Margins

Gain, phase, and complex margins are defined to describe the proximity of the Nyquist eigenloci to the $-1$ point. We define the multivariable gain and phase margins by injecting a complex equal diagonal perturbation diag $\{k_i e^{j\phi_i}\}$ at the plant input. We note that this is the same as perturbations at the plant output for our unity feedback system in Figure 1.

**Gain Margin (GM)**: Let $\phi_i = 0$ and $k_i = K$ be the minimum gain factor in all channels that drives the system unstable, $i = 1, 2, \cdots, m$. We may determine the multivariable GM as follows. From (4) we have that

$$\det\,[I + KL(s)] = \Pi_{i=1}^{i=m}(1 + K\lambda_i(s)) = 0, \tag{3}$$

resulting in

$$K = -\frac{1}{\lambda_i(j\omega)}, \tag{4}$$

or we find the *minimum* value of $K > 1$ for which the above equations (3)-(4) have a solution for some $\omega_{_{GM}}$

$$\mathrm{GM} = |K| = \left|\frac{1}{\lambda_i(j\omega_{_{GM}})}\right|. \tag{5}$$

Assume that the system has a nominal gain $K_{\text{nom}}$ and the Nyquist Stability analysis shows that the system is stable for $K_1 < K_{\text{nom}} < K_2$. Then the "upward gain margin" is

$$\mathrm{GM}_u = \frac{K_2}{K_{\text{nom}}}, \tag{6}$$

and the "downward gain margin" is:

$$\mathrm{GM}_d = \frac{K_1}{K_{\text{nom}}}. \tag{7}$$

It is customary to pick $K_{\text{nom}} = 1$ in stability margin analysis. These definitions are the same ones as in the SISO case [8].

**Phase Margin (PM)**: Let $k_i = K = 1$ and $\phi_i$ be the *minimum* phase lag in all channels that drives the system unstable, $i = 1, 2, \cdots, m$. We insert the complex diagonal perturbation $k_i e^{-j\phi_i} I$ in front of the loop gain. With $K = 1$

$$\det\,[I + KL(s)] = \Pi_{i=1}^{i=m}(1 + e^{-j\phi_i}\lambda_i(s)) = 0, \tag{8}$$

we have

$$e^{-j\phi_i}\lambda_i(j\omega) = -1, \tag{9}$$

and we find the *minimum* value of $\phi_i > 0$ for which the above equations (8)-(9) have a solution for some $\omega_{_{PM}}$

$$\mathrm{PM} = \phi_i = 180^\circ + \arg(\lambda_i(j\omega_{_{PM}})), \tag{10}$$

hence the Phase Margin (PM) is determined when the magnitude of $|\lambda_i(j\omega)|$ is unity. We compute the PM by measuring the angle formed by the real axis and the line going through the origin and the point where a circle of radius one centered at the origin intersects the Nyquist eigenloci. Again, this is the same as in the SISO case [8].

**Complex Margin (CM)**: O. J. M. Smith [9] introduced the idea of a vector margin for the SISO case. Because the vector margin is a single margin parameter, it removes all the ambiguities in assessing stability that comes with GM and PM in combination. In the past it has not been used extensively due to the difficulty in computing it. However, that barrier has been removed and the idea of using the vector margin [8] and the complex margin to describe the degree of stability is much more feasible now due to faster CPUs. Let $\alpha$ be the vector from the $-1$ point to the closest approach point of the Nyquist eigenloci. $\alpha$ may be determined as follows

$$|\alpha(\omega_1)| = \inf_{\omega}\ \min_{i} |1 + \lambda_i(KL(j\omega))|\,, i = 1, 2, \cdots, m. \tag{11}$$

We refer to $\alpha$ as the multivariable complex margin (CM) and the corresponding frequency as $\omega_1$. Note that $|\alpha|$ is the radius of the smallest circle with the center at $-1$ that is tangent to the Nyquist eigenloci. If the points of tangency are at $(a \pm jb)$, then the multivariable complex margin is given by

$$\alpha = -1 - (a \pm jb), \tag{12}$$

where

$$\lambda_i(j\omega_1) = a \pm jb, \tag{13}$$

and $\Delta = \alpha I$ is the minimum *additive* equal diagonal complex perturbation that drives the system unstable and $\Delta_1 = \alpha(KL)^{-1}$ is the associated *multiplicative* complex perturbation that drives the system unstable. The vector margin is the same as $\alpha$ for SISO systems [6].

Theorem 2: Consider the proper square $(m \times m)$ rational multivariable system with the unity feedback system with loop transfer function $KL(s)$ with no hidden unstable modes. Assume that the closed-loop system is stable for the nominal gain $K = K_{\text{nom}} = 1$ and also for the range of gains $K_1 < K_{\text{nom}} < K_2$. Then the upward gain margin is $\mathrm{GM}_u$ and the downward gain margin is $\mathrm{GM}_d$ as in Eqn. (6)-(7). The phase margin (PM) is given by Eq. (10). The complex margin $\alpha$ is given as in Eqn. (11-12). The minimum additive complex equal diagonal perturbation $\Delta = \alpha I$ drives the system unstable.

*Proof:* The addition of the gain factors of $\mathrm{GM}_u$ or $\mathrm{GM}_d$ to the loop gain, $L(s)$, will decrease/raise the gain of the system and will expand/contract the eigenloci of the product, $\mathrm{GM}_i L(s)$, and will force it to just pass through the $-1$ point resulting in sustained oscillations. Hence the closed-loop system will go unstable for any additional gain perturbation.

Similarly, the addition of the phase lag/phase advance (due to symmetry of the eigenloci) of PM degrees rotates each point on the Nyquist eigenloci resulting in a rotation of the entire eigenloci through an angle of PM degrees around the $-1$ point. With the nominal system being stable, such a rotation results in the Nyquist eigenloci passing through the $-1$ point leading to sustained oscillations. Instability will result with any additional phase perturbation. The additive complex equal diagonal perturbation of $\Delta = \alpha I$ will also result in the eigeloci of $(L(s) + \Delta)$ passing through the $-1$ point. Hence any larger additive complex equal diagonal perturbation will drive the system unstable. We can see that as follows. Compute an eigenvector (spectral) decomposition of $KL(j\omega)$ at each frequency $\omega$

$$KL(j\omega) = V(j\omega)\Lambda(j\omega)V^{-1}(j\omega), \tag{14}$$

where the columns of $V(s)$ are the eigenvectors of $KL(j\omega)$ and the eigenvalues are

$$\Lambda(j\omega) = \mathrm{diag}\{\lambda_1(j\omega), \lambda_2(j\omega), \cdots, \lambda_m(j\omega)\}, \tag{15}$$

where the $\lambda_i(s)$ are the eigenvalues and are assumed to be distinct for simplicity. Now consider forming the additive perturbation matrix

$$\Delta = -V(j\omega_1)\ \alpha I\ V^{-1}(j\omega_1), \tag{16}$$

with $\alpha$ chosen according to Eq. (12) then $KL(j\omega) + \Delta$ will have an eigenvalue at $-1$, hence the Nyquist eigenloci will go through the $-1$ point and the system will go unstable for any larger perturbation than $\Delta$. If the eigenvalues are repeated, we compute the Jordan form and the same reasoning still holds. ■

It is known that the frequency ($\omega_s$) at which the vector margin is computed for the SISO case corresponds to the peak value of the sensitivity function:

$$|\alpha| = \|S(j\omega)\|_{H_\infty}^{-1} = \left(\inf_{\omega} \|S(j\omega)\|\right)^{-1} = (\overline{\sigma}(S(\omega_s))^{-1}. \tag{17}$$

However, it is interesting to observe from our examples that this is generally *not* true in the MIMO case.

## IV. Examples

We will now illustrate the above ideas by applying them to several examples. Example 1 [7]: Consider the system

$$L(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix}. \tag{18}$$

The system has poles at $-1$, $-2$, and transmission zeros at $-1$, and $-2$. The eigenloci of $L(s)$ are given by

$$\det\,[\lambda I - L(s)] = \frac{1}{25}\frac{a(s)\lambda^2 + b(s)\lambda + 16}{(s+1)(s+2)} = 0 \tag{19}$$

$$a(s) = 25s^2 + 75s + 50, \qquad b(s) = -40s + 60,$$

and results in the two eigenvalues,

$$\lambda_1(s) = \frac{2}{5}\frac{(-3 + 2s + (1-24s)^{1/2})}{(s^2+3s+2)},$$

$$\lambda_2(s) = -\frac{2}{5}\frac{(3 - 2s + (1-24s)^{1/2})}{(s^2+3s+2)}.$$

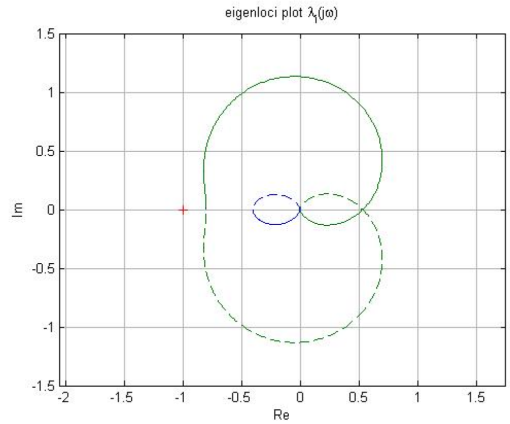


Fig. 2. Multivariable Nyquist eigenloci.

Note that these are not rational functions of $s$. The eigenloci is plotted in Figure 2 using the `nyqmimo` script from MATLAB Central. From the plot we see that for stability we must have

$$-\infty < -\frac{1}{K} < -0.8, -0.4 < -\frac{1}{K} < 0, 0.533 < -\frac{1}{K} < \infty. \tag{20}$$

The intersection of these inequalities leads to

$$-1.8762 < K < 1.25, \tag{21}$$

and

$$K > 2.5, \tag{22}$$

so $\mathrm{GM_u} = 1.25$. This is a *conditionally stable* system in that instability results if the nominal gain ($K_{\mathrm{nom}} = 1$) is lowered or raised and is very common in multivariable systems. Hence it will have multiple values for GM as seen above. From the plot we see that a circle with unit radius centered at the origin crosses the Nyquist eigenloci at $-0.7619 \pm j0.5873$ corresponding to $\omega = 0.2215$ rad/sec. This is where $|\lambda_2(j\omega)| = 1$ and results in the multivariable phase margin of PM $= \pm 37.62°$. The point of the closest approach of the Nyquist eigenloci to the $-1$ point is $-0.8$. Therefore the minimum additive perturbation that drives the system unstable is $\alpha\ \mathrm{I}_2 = -0.2\ \mathrm{I}_2$. Note that $\alpha = -1 - \lambda_2(j0)$ and in this case, the frequency for the closest approach to the $-1$ point ($\omega = 0$ rad/sec) does coincide with the frequency where the maximum singular value of the sensitivity function is a maximum ($\omega_s = 0$ rad/sec).

We now compare the norm of the additive perturbation found above ($||\alpha I||_2$) with the unstructured additive stability margin of [11]. Figure 3 shows a comparison of the two. We see that the complex margin has a larger magnitude at low frequencies. Next we compare the associated multiplicative perturbation with the unstructured multiplicative stability margin of [11] as shown in Figure 4. We again see that the complex margin has a higher magnitude at low frequencies.

Example 2 [8]: Consider the model of the laboratory RTP system [8] where

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \tag{23}$$

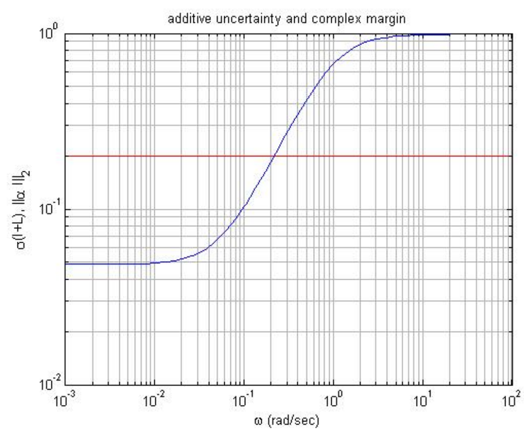


Fig. 3. Comparison of stability margins for additive perturbation.

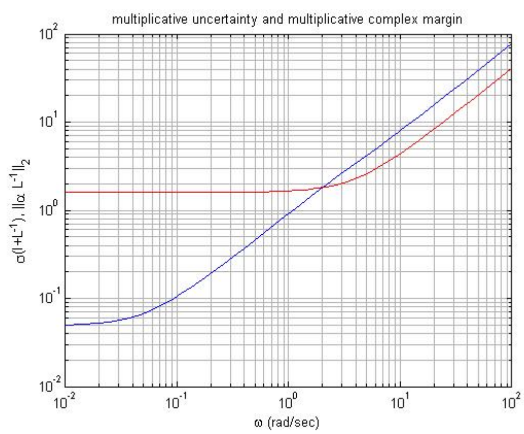


Fig. 4. Comparison of stability margins for multiplicative perturbation.

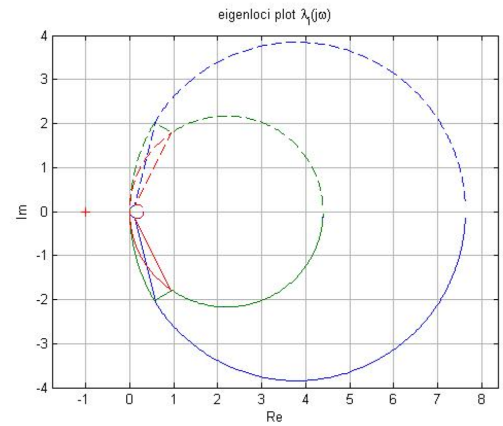


Fig. 5. Multivariable Nyquist eigenloci for RTP example.

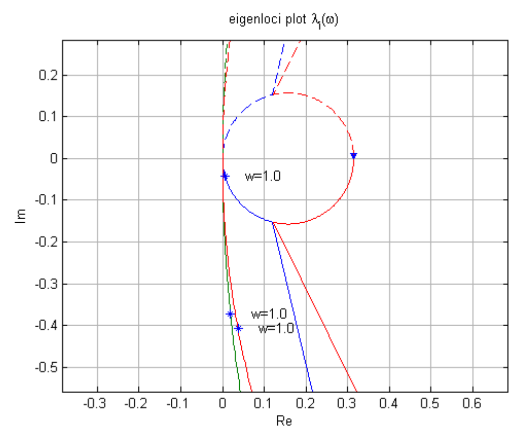


Fig. 6. Nyquist eigenloci for RTP example with $\omega$ values.

The three open-loop poles are located at $-0.0527$, $-0.0863$, and $-0.1482$ and there are no transmission zeros. The loop gain matrix is

$$L(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \tag{24}$$

The eigenloci plot is shown in Figure 5 and consists of three circles. It is interesting to notice the apparent “jump” in the three eigenloci circles at the frequency of $\omega = 0.14$ rad/sec. This can be explained if we plot the eigenloci vs. frequency. From such a plot we see that the first eigenvalue shown in blue switches order with the third in red. So indeed there is no discontinuity in the eigenloci curves and simply a reordering of the eigenvalues is occurring. From Figure 5 we see that for stability we must have

$$-\infty < -\frac{1}{K} < 0, \quad \text{and} \quad -\frac{1}{K} > 7.62. \tag{25}$$

The intersection of these inequalities leads to

$$K > -0.1312. \tag{26}$$

From the plot we see that a circle with unit radius centered at the origin crosses the Nyquist loci at $+0.0635 \pm j1$ corresponding to $\omega = 0.3746$ rad/sec. This is where $|\lambda_2(j\omega)| = 1$ and this results in the multivariable phase margin of PM $= \pm 93.63°$. The point of the closest approach of the Nyquist loci to the $-1$ point is the origin. Therefore the minimum additive perturbation that drives the system unstable is $\alpha\, \mathrm{I}_3 = -\mathrm{I}_3$. Note that $\alpha = -1 - \lambda_1(j\infty)$ and the frequency for the closest approach to the $-1$ point ($\omega = \infty$ rad/sec) does *not* coincide with the frequency where the maximum singular value of the sensitivity function is a maximum ($\omega_s = 1$ rad/sec) as shown in Figure 6.

We now compare the norm of the additive perturbation found above ($||\alpha I_3||_2$) with the unstructured additive stability margin of [11]. Figure 7 shows a comparison of the two. We see that the complex margin is lower in norm at low frequencies and comparable to the unstructured additive margin at high frequencies. Next we compare the associated multiplicative perturbation with the unstructured multiplicative stability margin of [11] as shown in Figure 8. We see that the complex margin has a higher magnitude at all frequencies.

To match the experimental data, we found out that the actual lamps provide 30% more power than we had initially modeled [6]. We may then form the scaling matrix

$$L_w = diag \begin{bmatrix} 1.3 & 1.3 & 1.3 \end{bmatrix}, \tag{27}$$

and recompute the robustness metrics with the new (scaled)

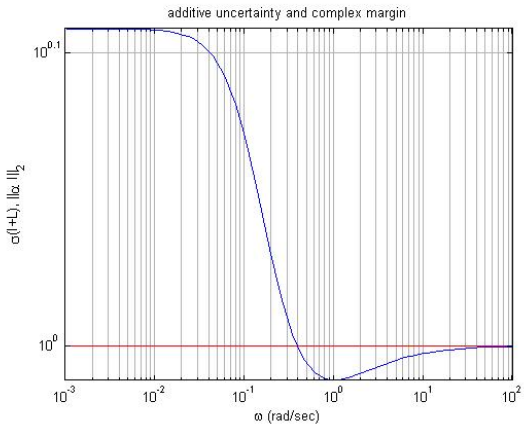

Fig. 7. Comparison of stability margins for additive perturbation.

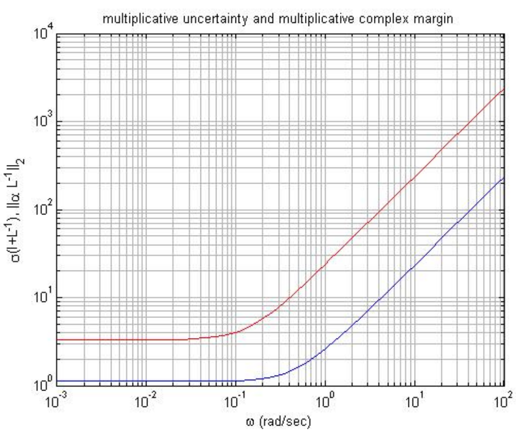

Fig. 8. Comparison of stability margins for multiplicative perturbation.

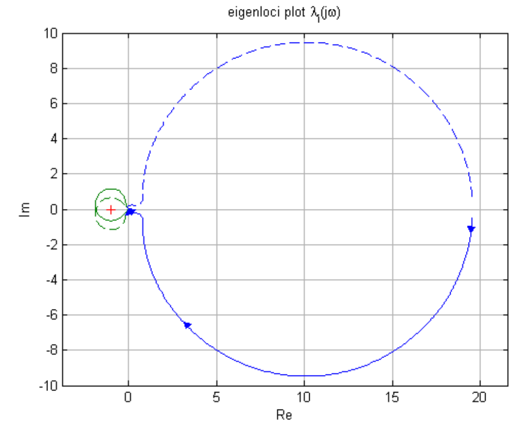

Fig. 9. Multivariable Nyquist eigenloci for drone example.

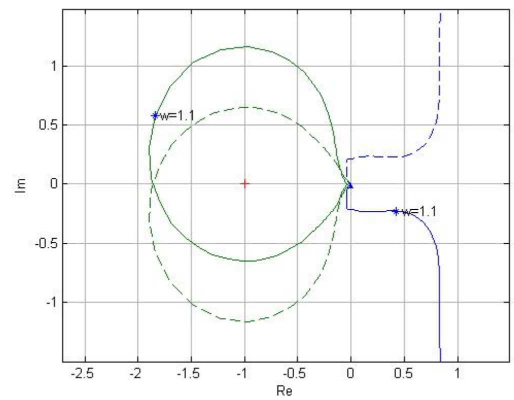

Fig. 10. Nyquist eigenloci for drone example with $\omega$ values.

loop gain matrix

$$\widetilde{L}(s) = L(s)L_w. \tag{28}$$

From the scaled eigenloci plot we find that for stability we must have

$$-\frac{1}{K} > 9.91, \tag{29}$$

that leads to

$$K > -0.1009, \tag{30}$$

as expected.

Example 3 [12, 13]: Consider the drone lateral attitude control system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \tag{31}$$

that has poles at $-0.0360$, $0.1884 \pm j1.0511$, $-3.2503$, $-20$, $-20$ with two poles in the RHP, and transmission zeros at 0, 0, and $-0.0671$. *Note that the system has an uncontrollable mode at the origin*. The loop gain matrix is

$$L(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \tag{32}$$

The eigenloci plot is shown in Figures 9 and has a rather complicated shape. Since the system has two open-loop poles in the RHP, $P = 2$ and $Z = P + N$. For stability we need $Z = 0$, so that $N = -2$. From the plot we see that for stability we must have

$$-\frac{1}{K} > -1.86, \quad \text{and} \quad -\frac{1}{K} < -0.0225. \tag{33}$$

The intersection of these inequalities leads to

$$0.5376 < K < 44.44, \tag{34}$$

so $\mathrm{GM_u} = 44.44$ and $\mathrm{GM_d} = 0.5376$. We see that a circle with unit radius centered at the origin crosses the Nyquist eigenloci plot at $-0.7778 \pm j0.6190$ corresponding to $\omega = 1.5783$ rad/sec. This is where $|\lambda_2(j\omega)| = 1$ and results in the multivariable phase margin of PM $= \pm 38.5°$. The point of the closest approach of the Nyquist loci to the $-1$ point is $-1 - 0.625j$. Therefore the minimum complex additive perturbation that drives the system unstable is $\alpha\,\mathrm{I}_2 = +0.625j\,\mathrm{I}_2$. Note that $\alpha = -1 - \lambda_2(j1.4526)$. The frequency for the closest approach to the $-1$ point ($\omega = 1.4526$ rad/sec) does *not* coincide with the frequency where the maximum singular value of the sensitivity function occurs ($\omega_s = 1.0915$ rad/sec), see Figure 10.

In contrast, the use of sensitivity and complementary sensitivity functions yields very conservative answers. The PM of 13.1° to 14.2° is computed. With proper scaling to reduce

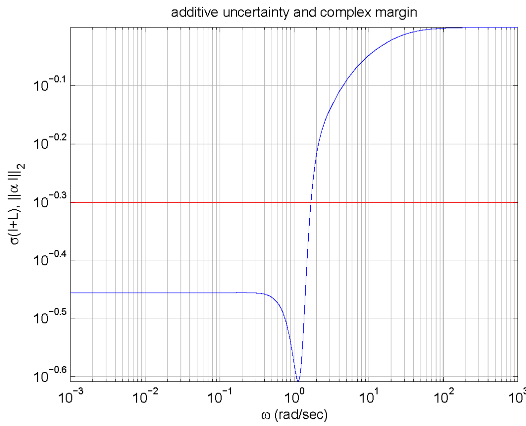


Fig. 11. Comparison of stability margins for additive perturbation.

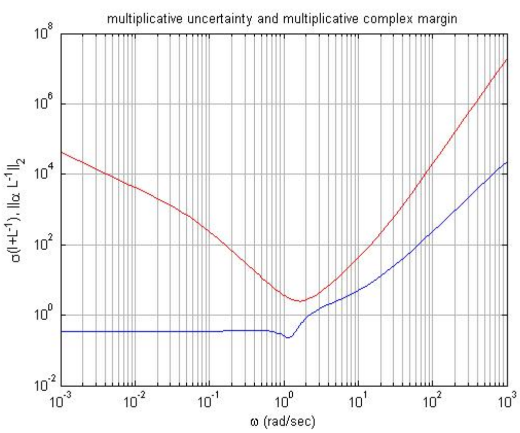


Fig. 12. Comparison of stability margins for multiplicative perturbation.

conservatism the answers are improved to $20.4°$ to $26.1°$, far short of the exact answer obtained above. Similarly, the GM margin computations are also very conservative. We find that $\mathrm{GM_d} = 0.7720$, $\mathrm{GM_u} = 1.228$, and $\mathrm{GM_d} = 0.8035$, $\mathrm{GM_u} = 1.3335$. With proper scaling these answers can be made less conservative $\mathrm{GM_d} = 0.645$, $\mathrm{GM_u} = 1.35$ and $\mathrm{GM_d} = 0.6918$, $\mathrm{GM_u} = 1.8197$. However, even the answers with scaling are still very conservative as compared to the exact answers obtained above.

We now compare the norm of the additive perturbation found above ($||\alpha I_2||_2$) with the unstructured additive stability margin of [11]. Figure 11 shows a comparison of the two. We see that the complex margin has a higher magnitude at low frequencies and a lower magnitude at high frequencies as compared to the unstructured perturbation. Next we compare the associated multiplicative perturbation with the unstructured multiplicative stability margin of [11] as shown in Figure 12. We see that the complex margin has a higher magnitude at all frequencies.

## V. Summary & Conclusions

The multivariable Nyquist eigenloci present a much richer family of plane curves as compared to the SISO case [2]. We have shown how to readily compute the exact values of the GM, PM, and the complex margin (CM) directly from the multivariable Nyquist eigenloci. This approach dismisses the computation of GM and PM from singular value plots of the sensitivity and complementary sensitivity functions that are generally very conservative. The exact equal diagonal complex additive perturbation that drives the system unstable can be computed directly from the complex margin. The effectiveness of our approach was demonstrated by application to several examples from process and aerospace control. Our approach needs to be used in combination with other more sophisticated robustness analysis approaches such as $\mu$, $k_m$ for a comprehensive robustness analysis.

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