The Shapes of Nyquist Plots

Connections with Classical Plane Curves

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The Nyquist criterion is a valuable design tool with applications to control systems and circuits [1], [2]. In this article, we show that many Nyquist plots are classical plane curves. Surprisingly, this connection seems to have gone unnoticed. We determine the precise shapes of several Nyquist curves and relate them to the shapes of the classical plane curves. Some classical plane curves are related to exactly proper or improper loop transfer functions, which do not roll off at high frequencies and thus are not physical.

Classical plane curves are used for robustness analysis in [3]. In addition, the area enclosed by the Nyquist curve is related to the Hilbert-Schmidt-Hankel norm of a linear system [4]. Therefore, knowledge of the precise shape of the Nyquist curve can provide additional useful information about the properties of a system.

The organization of this article is as follows. We first give a brief history of plane curves and then describe various plane curves. We then state some results that relate Nyquist plots to plane curves and present various illustrative examples. We end with some concluding remarks.

PLANE CURVES

Plane algebraic curves have been studied for more than 2000 years with applications to architecture, astronomy, and the arts [5]–[10]. Straight lines and circles were defined in antiquity, by Thales around 600 B.C., with applications to architecture. The classical mathematical problems in antiquity include the determination of π, the trisection of an angle, and the Delian problem, which concerns the amount that the side length of a cube needs to be increased to double its volume. All three problems are related to plane curves.

The cissoid of Diocles and the conchoid of Nicomedes were studied around 180 B.C. The Greeks used the cissoid of Diocles to attempt to solve the problem of trisecting an angle. The cissoid is the most ancient example of a curve with a cusp singularity. Conchoids were used in the construction of vertical columns, which are common in Greek, Roman, and Persian architecture. The discovery of conic sections in 350 B.C. resulted in the study of the intersection of cones with planes. Ellipses, parabolas, and hyperbolas were constructed around 150 B.C. by Menaechmus.

After a long intermission, starting with Dürer in 1525 and for the following 300 years during the Renaissance, there was tremendous interest in plane curves by the eminent mathematicians of the day, including Bernoulli, Euler, Huygens, Newton, Descartes, and Pascal. Kepler tried a variety of curves before settling on the ellipse as the best fit to the shape of planetary orbits. The invention of calculus in the second half of the 17th century had a strong influence on the study of curves. For example, the nephroid was shown by Huygens to be the solution to a classical optical problem, namely, it is the catacaustic of parallel light rays falling on a circle [10]. In 1696, Bernoulli posed a minimum-time optimal control problem whose solution, given the next day by Newton, is the brachistochrone, which is a section of a cycloid curve [11]. In mechanics, plane curves were applied to the design of gears and motors [10]. James Watt investigated Watt's curve, which is produced by a linkage of rods connecting two wheels of steam locomotives. Lissajous patterns were discovered in 1850 by the French physicist J.A. Lissajous with applications to electrical engineering and vibrations. The development of analytic and descriptive geometry in Europe was accelerated during the mid-19th century. Descartes led the investigation of curves in the complex projective plane. T.J. Freeth, an English mathematician, published a paper on strophoids in 1879.

Cardioid

The name cardioid, which means heart shaped, was used by de Castillon in *Philosophical Transactions of the Royal Society* in 1741 to refer to the curve shown in Figure 1 [6], [7]. The cardioid is given in polar coordinates by

$$r = 2a(1 + \cos\theta). \tag{1}$$

To express (1) in Cartesian coordinates we use the relations

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{2}$$

$$r = \sqrt{x^2 + y^2}, \tag{3}$$

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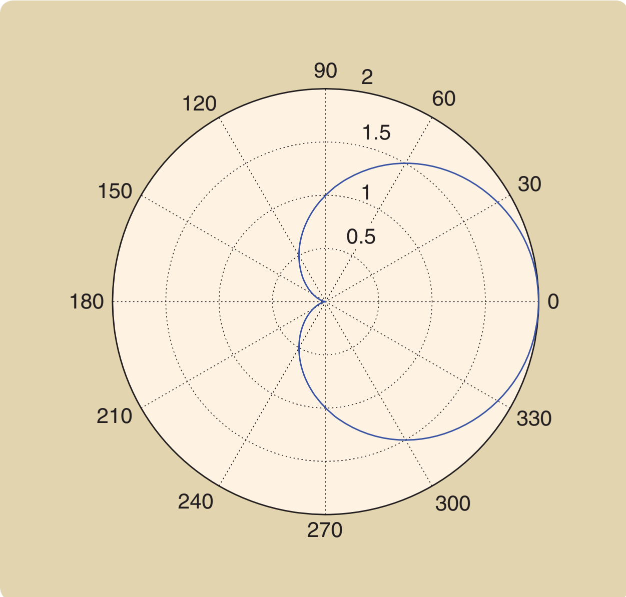

FIGURE 1 Plot of the cardioid with the polar equation $r = 2a(1+\cos\theta)$. The name cardioid, which means heart shaped, was first used by de Castillon in *Philosophical Transactions of the Royal Society* in 1741. This curve has a cusp at the origin.

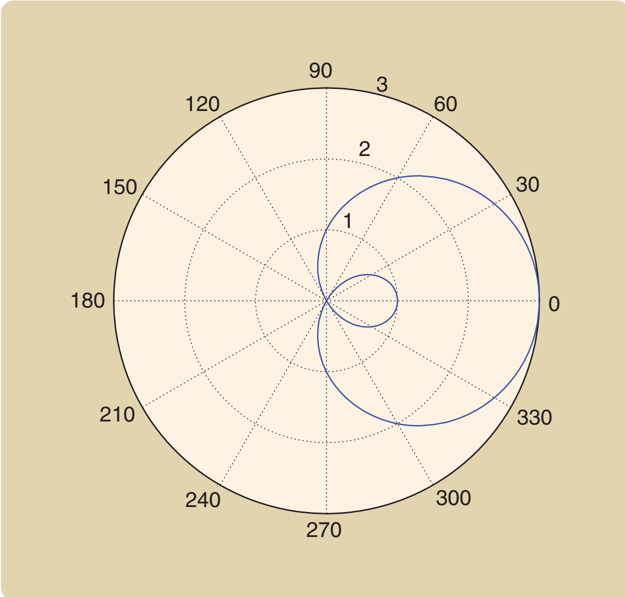

FIGURE 2 Plot of the limaçon with the polar equation $r = 2a\cos\theta + b$. The limaçon, which means "snail" in French and from the Latin limax, was first investigated by Dürer in 1525.

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \quad x \neq 0. \tag{4}$$

We rewrite (1) in the form

$$r - 2a\cos\theta = 2a. \tag{5}$$

Multiplying both sides of (5) by r and squaring both sides of the resulting equation and using (2)–(4) yields the quartic equation

$$(x^2 + y^2 - 2ax)^2 = 4a^2(x^2 + y^2). \tag{6}$$

The area enclosed by the cardioid is [7]

$$A = 6\pi a^2. \tag{7}$$

Limaçon

The limaçon, whose name means snail in French from the Latin word limax, was first investigated by Dürer in 1525, who gave a method for drawing the curve [6], [7]. The curve was rediscovered by Étienne Pascal, father of Blaise Pascal, and named by Gilles-Personne Roberval in 1650. This curve, which is shown in Figure 2, is described by the polar equation

$$r = 2a\cos\theta + b. \tag{8}$$

For details, see "Dad, That Is a Limaçon." If $|2a| = |b|$, then the limaçon becomes a cardioid. If $|2a| < |b|$, then the limaçon has an inner loop. At points on the inner loop corresponding to the values $120° < \theta < 240°$, r becomes negative. Note that [5] in polar coordinates the point (r, θ), where $r < 0$, denotes the point $(|r|, \theta + \pi)$. The size of the inner loop decreases as $|2a/b|$ decreases. If $|2a| < |b|$, then the limaçon has no inner loop. For $1/2 < |2a/b| < 1$, the limaçon's cusp is smoothed and becomes a dimple. The limaçon loses its dimple when $|2a/b| = 1/2$.

To express the limaçon in Cartesian coordinates, we multiply both sides of (8) by r and rearrange terms to obtain

$$r^2 - 2ar\cos\theta = br. \tag{9}$$

By squaring both sides of (9) and using (2)–(4) we obtain

$$(x^2 + y^2 - 2ax)^2 = b^2(x^2 + y^2). \tag{10}$$

The area enclosed by the limaçon is given by [7]

$$A = \begin{cases} (2a^2 + b^2)\pi, & b \geq 2a, \\ (2a^2 + b^2)\left(\pi - \cos^{-1}\dfrac{b}{2a}\right) + \dfrac{3}{2}b\sqrt{4a^2 - b^2}, & b < 2a. \end{cases} \tag{11}$$

Dad, That Is a Limaçon

I was drawing the curve in Figure 2 on our home computer. My 17-year-old son looked over my shoulder and said: "Dad, that is a limaçon." I was very surprised and asked: "How do you know?" He said, "Oh, we plotted that two years ago in my sophomore trigonometry class." I knew right then that this article had to be written!

Trisectrix

The combination of a compass and a straightedge cannot be used to trisect an arbitrary angle. However, a form of the limaçon can be used to trisect an angle. If $a = b$ in (10), then the curve shown in Figure S1 is called a *trisectrix* and satisfies $\angle OAB = (1/3)\angle ABC$. Therefore, it can be used to trisect an angle.

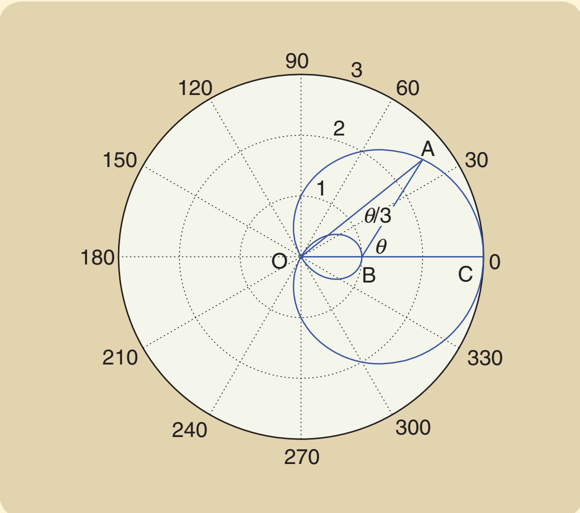

FIGURE S1 Illustration of the trisectrix plane curve. A *trisectrix* is a special limaçon that can be used to trisect an angle. The trisectrix of Maclaurin can also be used to trisect an angle as shown in Figure S3.

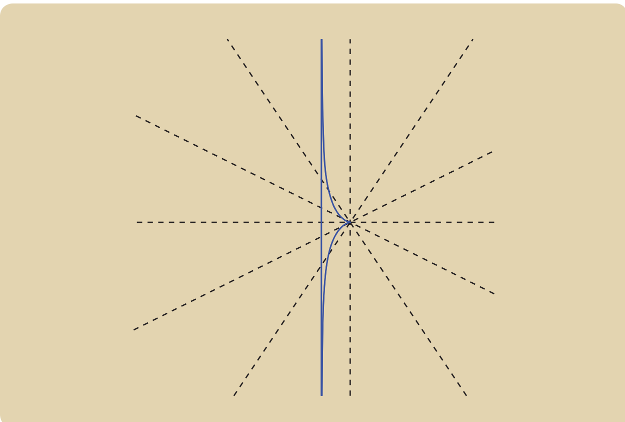

FIGURE 3 Plot of the cissoid of Diocles with the polar equation $r = -a\,(\sin^2\theta/\cos\theta)$. This curve, which means ivy shaped, has the asymptote $x = a$ and a single cusp.

If $a = b$, then the limaçon is called a *trisectrix*, which can be used to trisect an angle. For details, see "Trisectrix."

Cissoid of Diocles

The cissoid of Diocles is named after the Greek mathematician Diocles who used it in 180 B.C. to solve the Delian problem mentioned above. A cissoid of Diocles, whose name means ivy shaped, is an unbounded plane curve with a single cusp that is symmetric about the line of tangency of the cusp as shown in Figure 3 [6], [7]. The pair of symmetric branches approach the same asymptote but from opposite directions. The polar equation is given by

$$r = -a\frac{\sin^2\theta}{\cos\theta}. \tag{12}$$

To express the cissoid of Diocles in Cartesian coordinates, we rewrite (12) as

$$r = -a\frac{y^2}{xr}. \tag{13}$$

Multiplying both sides of (13) by r and substituting from (2)–(4) yields

$$x^3 = -y^2(a + x). \tag{14}$$

The cissoid of Diocles has the asymptote $x = a$.

Strophoid

The strophoid, investigated by Barrow in 1670, is the plane curve shown in Figure 4. The word "strophoid" means a belt with a twist. The strophoid is given by the polar equation

$$r = a(\cos 2\theta)\sec\theta. \tag{15}$$

To derive the Cartesian form of (15), rewrite (15) as

$$r = a(2\cos^2\theta - 1)\sec\theta. \tag{16}$$

Squaring both sides of (16) and substituting from (2)–(4) yields

$$y^2 = x^2\frac{a - x}{a + x}. \tag{17}$$

The strophoid has an asymptote given by $x = -a$.

Cayley's Sextic

Cayley's sextic was discovered by Maclaurin in 1718 but studied in detail by Cayley [7]. This curve, which is shown in Figure 5, is described by the polar equation

$$r = 4a\cos^3\frac{\theta}{3}. \tag{18}$$

To derive the Cartesian form, we first rewrite (18) as

$$r = 4a\left(\frac{\cos\theta + 3\cos\frac{\theta}{3}}{4}\right). \tag{19}$$

Multiplying both sides of (19) by r and rearranging yields

$$r^2 - ar\cos\theta = 3ar\cos\frac{\theta}{3}. \tag{20}$$

Cubing both sides of (20) and using (2)–(4) and (18) leads to

$$4(x^2 + y^2 - ax)^3 = 27a^2(x^2 + y^2)^2. \tag{21}$$

Folium of Kepler

The folium of Kepler studied by Kepler in 1609 is the leaf-shaped plane curve with the polar equation

$$r = (\cos\theta)(4a\sin^2\theta - b). \tag{22}$$

To express (22) in Cartesian coordinates, we use (2)–(4) to obtain

$$r = \frac{x}{r}\left(4a\frac{y^2}{r^2} - b\right). \tag{23}$$

Multiplying both sides of (23) by r and again using (2)–(4) leads to

$$(x^2 + y^2)[x(x + b) + y^2] - 4axy^2 = 0. \tag{24}$$

If $b \geq 4a$, the curve has only one folium or leaf. Otherwise, the curve has more than one leaf. Figure 6 shows Kepler's folium for the case $a = 1$ and $b = 4$.

Nephroid

The nephroid, meaning kidney shaped, was studied by Huygens in 1678. This shape is described by the polar equation

$$r^2 = \frac{1}{2}a^2(5 - 3\cos 2\theta), \tag{25}$$

which has two cusps. In Cartesian variables, the nephroid is described by

$$x = a\left(3\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right), \tag{26}$$

$$y = a\left(3\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right) = 4a\sin^3\frac{\theta}{2}. \tag{27}$$

Cubing both sides of (25) and using (26)–(27) and (2)–(4) yields

$$(x^2 + y^2 - 4a^2)^3 - 108a^4y^2 = 0. \tag{28}$$

Figure 7 illustrates the nephroid for $a = 1$.

Nephroid of Freeth

The nephroid of Freeth, which is shown in Figure 8, is described by the polar equation

$$r = a\left(1 + 2\sin\frac{\theta}{2}\right), \quad a > 0. \tag{29}$$

Rearranging the terms in (29) and squaring both sides yields

$$(r - a)^2 = 2a^2(1 - \cos\theta). \tag{30}$$

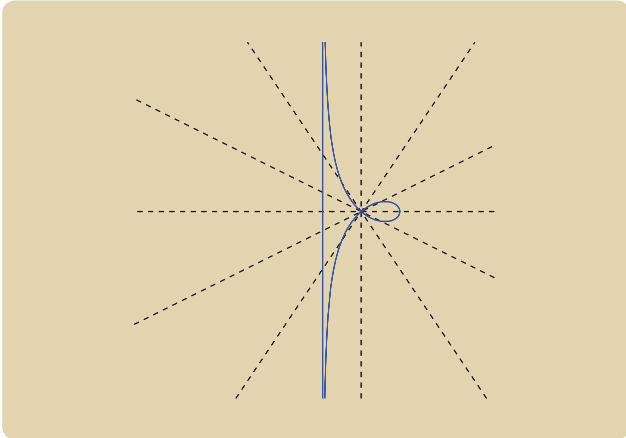

FIGURE 4 Plot of the strophoid with the polar equation $r = 2a(\cos 2\theta)\sec\theta$. This curve, which means shaped like a belt with a twist, was investigated by Barrow in 1670.

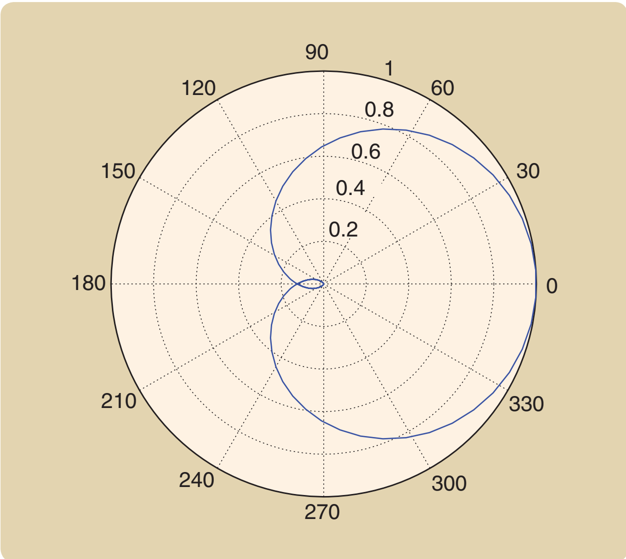

FIGURE 5 Plot of Cayley's sextic with the polar equation $r = 4a\cos^3(\theta/3)$. This curve, which resembles a shifted limaçon, was discovered by Maclaurin in 1718, but studied in detail by Cayley.

Expanding the left-hand side, rearranging, and multiplying both sides of (30) by r leads to

$$r(r^2 - a^2) = 2a(r^2 - ax). \tag{31}$$

Now squaring both sides of (31) and using (2)–(4) leads to

$$(x^2 + y^2)(x^2 + y^2 - a^2)^2 - 4a^2(x^2 + y^2 - ax)^2 = 0. \tag{32}$$

This curve is distinct from the nephroid.

Shifted Plane Curves

Shifted versions of plane curves can be obtained by replacing x and y by $x - x_0$ and $y - y_0$, respectively. For example,

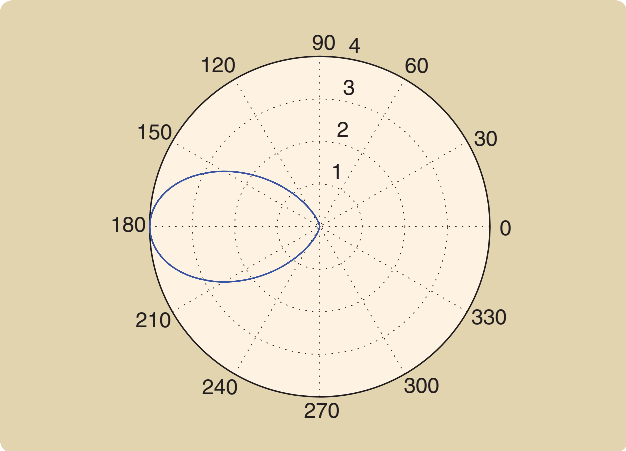

FIGURE 6 Plot of the folium of Kepler with the polar equation $r = (\cos\theta)(4a\sin^2\theta - b)$. The folium of Kepler, which means leaf shaped, was studied by Kepler in 1609.

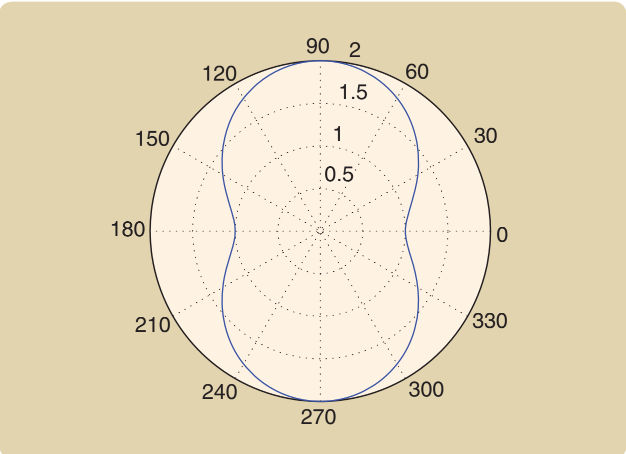

FIGURE 7 Plot of the nephroid with the polar equation is $r^2 = (1/2)\,a^2(5 - 3\cos\theta)$. The nephroid, which means kidney shaped, was studied by Huygens in 1678. This curve has two cusps.

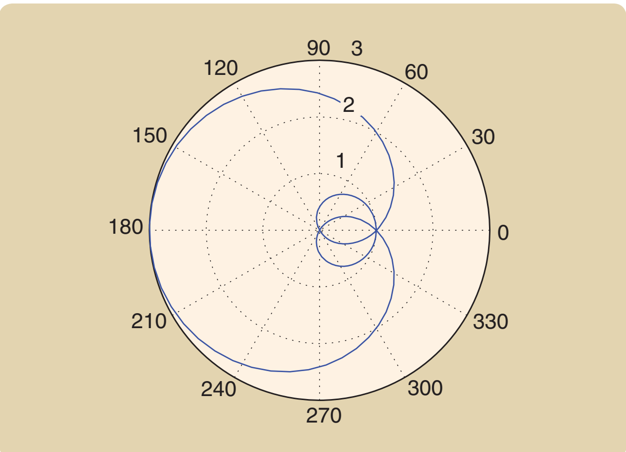

FIGURE 8 Plot of the nephroid of Freeth with the polar equation $r = a(1 + 2\sin(\theta/2))$. This curve was studied in 1879 by the English mathematician T.J. Freeth.

the Cartesian equation for the shifted nephroid of Freeth (32) is given by

$$((x - x_0)^2 + (y - y_0)^2)((x - x_0)^2 + (y - y_0)^2 - a^2)^2 - 4a^2((x - x_0)^2 + (y - y_0)^2 - a(x - x_0))^2 = 0. \quad (33)$$

NYQUIST CURVES

In this section, we relate the shapes of various Nyquist plots to the plane curves presented in the previous section.

Theorem 1

Consider the second-order loop transfer function

$$L(s) = \frac{1}{(s + \alpha)(s + \beta)}, \quad (34)$$

where $\alpha > 0$ and $\beta > 0$. Then the Nyquist plot of $L(s)$ is the cardioid

$$x^4 + y^4 - \frac{1}{\alpha\beta}x^3 + 2x^2y^2 - \frac{1}{\alpha\beta}xy^2 - \frac{1}{\alpha\beta(\alpha + \beta)^2}y^2 = 0. \quad (35)$$

Proof

For $\omega > 0$ and $s = \jmath\omega$, we have

$$\begin{aligned} L(\jmath\omega) &= \frac{1}{(\jmath\omega + \alpha)(\jmath\omega + \beta)} \\ &= \frac{1}{\sqrt{\omega^2 + \alpha^2}\sqrt{\omega^2 + \beta^2}} e^{-\jmath(\theta_1(\omega) + \theta_2(\omega))} \\ &= \frac{1}{\sqrt{\omega^2 + \alpha^2}\sqrt{\omega^2 + \beta^2}} (\cos(\theta_1(\omega) + \theta_2(\omega)) \\ &\quad - \jmath\sin(\theta_1(\omega) + \theta_2(\omega))), \end{aligned}$$

where

$$\theta_1(\omega) = \tan^{-1}\left(\frac{\omega}{\alpha}\right), \quad \theta_2(\omega) = \tan^{-1}\left(\frac{\omega}{\beta}\right),$$

which leads to the relation

$$\frac{y}{x} = -\tan(\theta_1(\omega) + \theta_2(\omega)) = -\frac{\frac{\omega}{\alpha} + \frac{\omega}{\beta}}{1 - \frac{\omega\omega}{\alpha\beta}} = -\frac{(\alpha + \beta)\omega}{\alpha\beta - \omega^2}. \quad (36)$$

Rewriting (36) as the quadratic equation

$$\omega^2 y - (\alpha + \beta)x\omega - \alpha\beta y = 0$$

yields

$$\omega = \frac{(\alpha + \beta)x \pm \sqrt{(\alpha + \beta)^2 x^2 + 4\alpha\beta y^2}}{2y}. \quad (37)$$

Furthermore, we have that

$$\mathrm{Re}(L(j\omega)) = x = -\frac{(\alpha+\beta)\omega}{(-\omega^2+\alpha\beta)^2+\omega^2(\alpha+\beta)^2}. \tag{38}$$

Substituting (37) into (38) yields (35). □

Example 1: Cardioid

Consider the loop transfer function [2]

$$L(s) = \frac{1}{(s+1)^2}. \tag{39}$$

It follows from Theorem 1 that the Nyquist plot of $L(s)$ is a cardioid. In polar coordinates we have that

$$r(\omega) = \frac{1}{\omega^2+1} = \frac{1}{2}(1+\cos 2\theta(\omega)),$$

where the Nyquist plot is shown in Figure 9 with the cusp point at the origin. The Cartesian equation is given by

$$\left(x^2+y^2-\frac{1}{2}x\right)^2 = \frac{1}{4}(x^2+y^2). \tag{40}$$

■

Theorem 2

Consider the proper second-order system with the loop transfer function with imaginary zeros given by

$$L(s) = \frac{s^2+\gamma}{(s+\alpha)(s+\beta)}, \tag{41}$$

where $\alpha > 0$, $\beta > 0$, and $\gamma > 0$. Then the Nyquist plot of $L(s)$ is the limaçon

$$\begin{aligned}(\beta^3+\beta+2\beta^2)x^4+(\gamma-2\beta^2-\beta^3-\beta-\gamma\beta^2-2\gamma\beta)x^3+\\(2\beta y^2+2\beta^3y^2+4\beta^2y^2+\gamma\beta^2+2\gamma\beta+\gamma)x^2+\\(-\gamma y^2-\beta^3y^2-2\beta^2y^2-2\gamma\beta y^2-\gamma\beta^2y^2-\beta y^2)x+\\\beta^3y^4+\beta y^4-\beta^2y^2+2\beta^2y^4-\gamma^2y^2+2\gamma\beta y^2=0.\end{aligned} \tag{42}$$

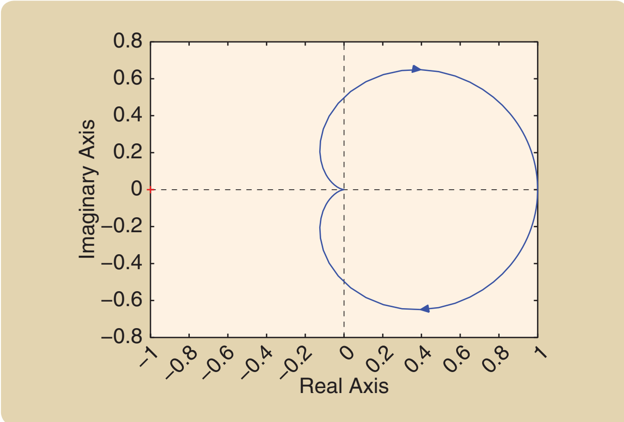

FIGURE 9 The Nyquist plot for the second-order loop transfer function $L(s) = 1/(s+1)^2$. The right-half plane is mapped into the inside of the cardioid. The polar equation is $r(\omega) = 0.5(1+\cos 2\theta(\omega))$, and the Cartesian equation is $(x^2+y^2-0.5x)^2 = 0.25(x^2+y^2)$.

Proof

For $s = j\omega$, we have the equation at the bottom of the page where

$$\theta_1(\omega) = \tan^{-1}\left(\frac{\omega}{\alpha}\right), \quad \theta_2(\omega) = \tan^{-1}\left(\frac{\omega}{\beta}\right).$$

Furthermore,

$$\frac{y}{x} = -\tan(\theta_1(\omega)+\theta_2(\omega)) = \frac{-(\alpha+\beta)\omega}{1-\alpha\beta\omega^2}. \tag{43}$$

Solving (43) for ω yields

$$\omega = \frac{(\alpha+\beta)x \pm \sqrt{(\alpha+\beta)^2x^2+4\alpha\beta y^2}}{2\alpha\beta y}. \tag{44}$$

Furthermore, we have the relations

$$r(\omega) = \sqrt{x^2+y^2} = -\frac{(-\omega^2+\gamma)}{\sqrt{\omega^2+\alpha^2}\sqrt{\omega^2+\beta^2}}, \tag{45}$$

$$L(j\omega) = \frac{-\omega^2+\gamma}{(j\omega+\alpha)(j\omega+\beta)}$$

$$= \begin{cases} \dfrac{(-\omega^2+\gamma)}{\sqrt{\omega^2+\alpha^2}\sqrt{\omega^2+\beta^2}} e^{-j(\theta_1(\omega)+\theta_2(\omega))} \\ = r(\omega)(\cos(\theta_1(\omega)+\theta_2(\omega)) - j\sin(\theta_1(\omega)+\theta_2(\omega))), & 0 < \omega < \sqrt{\gamma}, \\ \dfrac{(-\omega^2+\gamma)}{\sqrt{\omega^2+\alpha^2}\sqrt{\omega^2+\beta^2}} e^{-j(\theta_1(\omega)+\theta_2(\omega)-\pi)} \\ = r(\omega)(-\cos(\theta_1(\omega)+\theta_2(\omega)) + j\sin(\theta_1(\omega)+\theta_2(\omega))), & \omega > \sqrt{\gamma}, \end{cases}$$

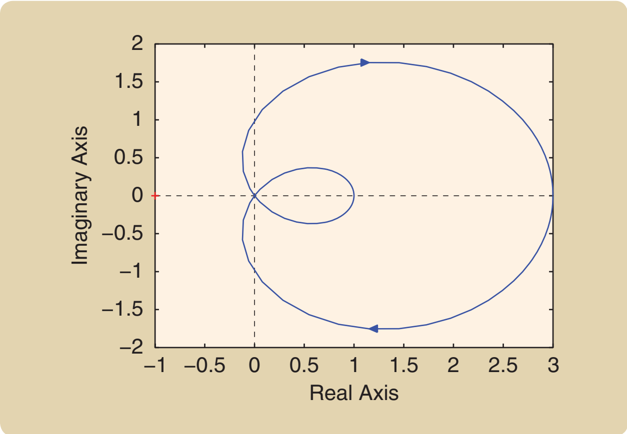

FIGURE 10 The Nyquist plot for the second-order loop transfer function $L(s)=(s^2+3)/(s+1)^2$. This limaçon has the polar equation $r(\omega)=1+\cos 2\theta(\omega)$, and the Cartesian equation $(x^2+y^2-2x)^2=x^2+y^2$.

$$\mathrm{Re}(L(\jmath\omega))=x=\frac{(-\omega^2+\gamma)(-\omega^2+\alpha\beta)}{(-\omega^2+\alpha\beta)^2+(\alpha+\beta)^2\omega^2}. \qquad (46)$$

Substituting for ω from (44) in (45) results in the polynomial Cartesian equation (42), which is the limaçon. □

Example 2: Limaçon

Consider the exactly proper [13] loop transfer function

$$L(s)=\frac{s^2+3}{(s+1)^2}. \qquad (47)$$

It follows from Theorem 2 that the Nyquist plot of $L(s)$ is a limaçon. In polar coordinates we obtain

$$r(\omega)=\frac{-\omega^2+3}{\omega^2+1}=1+\cos 2\theta(\omega).$$

The Nyquist plot is shown in Figure 10 with the cusp point at the origin. The Cartesian equation is

$$(x^2+y^2-2x)^2=x^2+y^2. \qquad (48)$$

■

Theorem 3

Consider the second-order Type I loop transfer function

$$L(s)=\frac{1}{s(s+\alpha)}, \qquad (49)$$

where $\alpha\neq 0$. Then the Nyquist plot of $L(s)$ is the cissoid of Diocles

$$x^3=-y^2\left(\frac{1}{\alpha^2}+x\right). \qquad (50)$$

Proof

We first consider the case $\alpha>0$. For $s=\jmath\omega$, we have

$$\begin{aligned}L(\jmath\omega)&=\frac{1}{\jmath\omega(\jmath\omega+\alpha)}\\&=\frac{1}{\omega\sqrt{\omega^2+\alpha^2}}e^{-\jmath\left(\frac{\pi}{2}+\theta(\omega)\right)}\\&=\frac{1}{\omega\sqrt{\omega^2+\alpha^2}}(-\sin\theta(\omega)-\jmath\cos\theta(\omega)),\end{aligned}$$

where

$$\theta(\omega)=\tan^{-1}\left(\frac{\omega}{\alpha}\right),\quad \frac{x}{y}=\tan\theta(\omega)=\frac{\omega}{\alpha},\quad \omega=\frac{\alpha x}{y}. \qquad (51)$$

Furthermore, we have

$$r(\omega)=\sqrt{x^2+y^2}=-\frac{1}{\omega\sqrt{\omega^2+\alpha^2}}. \qquad (52)$$

Substituting for ω from (51) in (52) results in (50).

We now consider the case $\alpha<0$. For $s=\jmath\omega$, we have

$$\begin{aligned}L(\jmath\omega)&=\frac{1}{\jmath\omega(\jmath\omega+\alpha)}\\&=\frac{1}{\omega\sqrt{\omega^2+\alpha^2}}e^{-\jmath\left(\frac{3\pi}{2}-\theta(\omega)\right)}.\\&=\frac{1}{\omega\sqrt{\omega^2+\alpha^2}}(-\sin\theta(\omega)+\jmath\cos\theta(\omega)),\end{aligned}$$

where

$$\theta(\omega)=\tan^{-1}\left(\frac{\omega}{\alpha}\right),\quad \frac{x}{y}=-\tan\theta(\omega)=-\frac{\omega}{\alpha},\quad \omega=-\frac{\alpha x}{y}, \qquad (53)$$

which leads to

$$r(\omega)=\sqrt{x^2+y^2}=\frac{1}{\omega\sqrt{\omega^2+\alpha^2}}. \qquad (54)$$

Substituting for ω from (53) into (54) yields (50). □

Example 3: Cissoid of Diocles

Consider the loop transfer function

$$L(s)=\frac{1}{s(s+1)}. \qquad (55)$$

It follows from Theorem 3 that the Nyquist plot of $L(s)$ is a cissoid of Diocles. In polar coordinates we have

$$r(\omega)=\frac{1}{\omega(1+\omega^2)^{1/2}}=\frac{1}{\tan\theta(\omega)(1+\tan^2\theta(\omega))^{1/2}}=\frac{\cos^2\theta(\omega)}{\sin\theta(\omega)}.$$

The Nyquist plot is shown in Figure 11, and the corresponding Cartesian equation is

$$x^3 = y^2(1-x). \tag{56}$$

■

Example 4: Cissoid of Diocles for an Improper System

Consider the improper loop transfer function

$$L(s) = \frac{s^2}{s+1}. \tag{57}$$

For $s = \jmath\omega$ and $\omega > 0$, we have

$$\begin{aligned} L(\jmath\omega) &= \frac{-\omega^2}{\jmath\omega+1} = \frac{-\omega^2}{(1+\omega^2)^{1/2}} e^{-\jmath\theta(\omega)} \\ &= r(\omega)(\cos\theta(\omega) - \jmath\sin\theta(\omega)). \end{aligned}$$

In polar coordinates, we obtain

$$r(\omega) = \frac{-\omega^2}{(1+\omega^2)^{1/2}} = -\sin\theta(\omega)\tan\theta(\omega).$$

Furthermore, we have

$$\frac{y}{x} = -\tan\theta(\omega).$$

The Nyquist plot is the cissoid of Diocles shown in Figure 12. The Cartesian equation is

$$x^3 = -y^2(1+x). \tag{58}$$

■

Theorem 4

Consider the third-order Type I loop transfer function

$$L(s) = \frac{1}{s(s+\alpha)(s+\beta)},$$

where $\alpha > 0$, $\beta > 0$. Then the Nyquist plot of $L(s)$ is the shifted strophoid

$$\begin{aligned} &3\alpha^2\beta y^4 + 2\alpha^2\beta x^4 + 5\alpha^2\beta x^2y^2 + \alpha^3x^2y^2 + \alpha^3y^4 + \beta^3y^4 + \\ &\alpha^4\beta^2x^5 + 2\alpha\beta^2x^4 + 3\alpha\beta^2y^4 + 5\alpha\beta^2x^2y^2 + \alpha^2\beta^4xy^4 + \\ &2\alpha^2\beta^4x^3y^2 + \alpha^4\beta^2xy^4 + x^3 + 2\alpha^3\beta^3xy^4 + 4\alpha^3\beta^3x^3y^2 + \\ &2\alpha^4\beta^2x^3y^2 + \alpha^2\beta^4x^5 + \beta^3y^2x^2 + 2\alpha^3\beta^3x^5 = 0. \end{aligned} \tag{59}$$

Proof

For $s = \jmath\omega$ and $\omega > 0$, we have

$$\begin{aligned} L(\jmath\omega) &= \frac{1}{\jmath\omega(\jmath\omega+\alpha)(\jmath\omega+\beta)} \\ &= \frac{1}{\omega\sqrt{\omega^2+\alpha^2}\sqrt{\omega^2+\beta^2}} e^{-\jmath(\frac{\pi}{2}+\theta_1(\omega)+\theta_2(\omega))} \\ &= \frac{1}{\omega\sqrt{\omega^2+\alpha^2}\sqrt{\omega^2+\beta^2}}(-\sin(\theta_1(\omega)+\theta_2(\omega)) \\ &\quad - \jmath\cos(\theta_1(\omega)+\theta_2(\omega))), \end{aligned}$$

where

$$\theta_1(\omega) = \tan^{-1}\left(\frac{\omega}{\alpha}\right), \quad \theta_2(\omega) = \tan^{-1}\left(\frac{\omega}{\beta}\right). \tag{60}$$

Moreover, we have

$$\frac{x}{y} = \tan(\theta_1(\omega)+\theta_2(\omega)) = \frac{\frac{\omega}{\alpha}+\frac{\omega}{\beta}}{1-\frac{\omega}{\alpha}\frac{\omega}{\beta}} = \frac{(\alpha+\beta)\omega}{\alpha\beta-\omega^2}. \tag{61}$$

We rearrange (61) to obtain

$$x\omega^2 + (\alpha+\beta)y\omega - \alpha\beta x = 0, \tag{62}$$

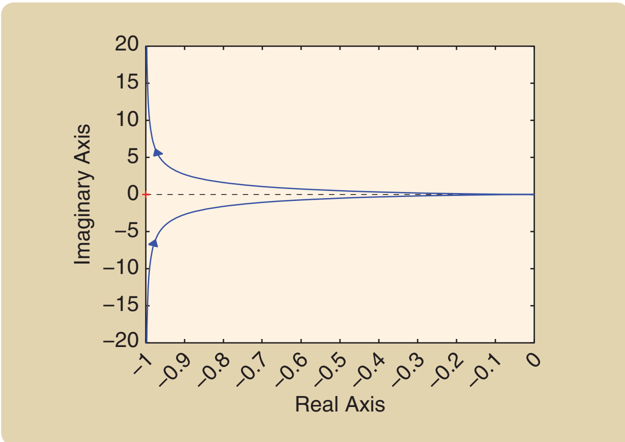

FIGURE 11 The Nyquist plot for the second-order Type I loop transfer function $L(s) = 1/[s(s+1)]$. This curve, which is a cissoid of Diocles has an asymptote at -1. The polar equation is $r(\omega) = \cos^2\theta(\omega)/\sin\theta(\omega)$, and Cartesian equation is $x^3 = y^2(1-x)$.

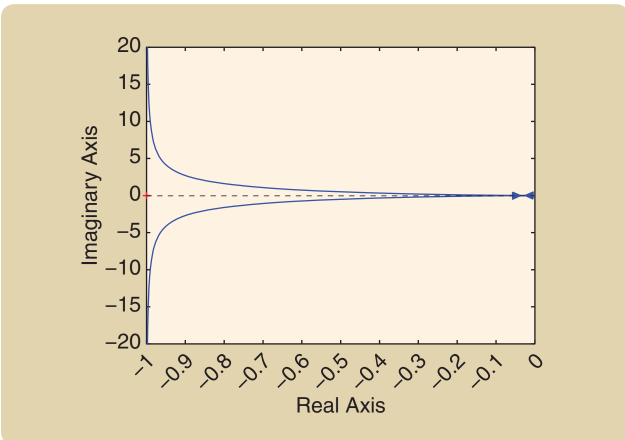

FIGURE 12 The Nyquist plot for the first-order improper loop transfer function $L(s) = s^2/(s+1)$. This curve, which is a cissoid of Diocles, has an asymptote at -1. The polar equation is $r(\omega) = -(\sin\omega)\tan\omega$, and the Cartesian equation is $x^3 = -y^2(1+x)$.

whose solution is

$$\omega = \frac{-(\alpha+\beta)y \pm \sqrt{(\alpha+\beta)^2 y^2 + 4\alpha\beta x^2}}{2x}. \tag{63}$$

We also see that

$$r(\omega) = \sqrt{x^2+y^2} = -\frac{1}{\omega\sqrt{\omega^2+\alpha^2}\sqrt{\omega^2+\beta^2}}. \tag{64}$$

Substituting for ω from (63) into (64) yields (59). □

Example 5: Shifted Strophoid

Consider the loop transfer function [2]

$$L(s) = \frac{1}{s(s+1)^2}. \tag{65}$$

It follows from Theorem 4 that the Nyquist plot of $L(s)$ is the shifted strophoid. The polar equation is

$$r(\omega) = \frac{1}{\omega(1+\omega^2)} = \frac{1+\cos 2\theta(\omega)}{2\tan\theta(\omega)}.$$

The Nyquist plot is shown in Figure 13, and the corresponding Cartesian equation is

$$4y^4x + 8x^3y^2 + 12x^2y^2 + 8y^4 + 4x^5 + 4x^4 + x^3 = 0. \tag{66}$$

■

Example 6: "Shifted Strophoid"

Consider the loop transfer function [2]

$$L(s) = \frac{s+1}{s\left(\frac{s}{10}-1\right)}. \tag{67}$$

For $s = \jmath\omega$ and $\omega > 0$, we have

$$\begin{aligned} L(\jmath\omega) &= \frac{10(\jmath\omega+1)}{\jmath\omega(\jmath\omega-10)} = \frac{10(1+\omega^2)^{1/2}}{\omega(1+\omega^2)} e^{-\jmath\left(\theta_1(\omega)-\theta_2(\omega)-\frac{\pi}{2}\right)} \\ &= r(\omega)(\sin(\theta_1(\omega)-\theta_2(\omega)) - \jmath\cos(\theta_1(\omega)-\theta_2(\omega))), \end{aligned}$$

where

$$\theta_1(\omega) = \operatorname{atan2}(\omega, 1),\ \theta_2(\omega) = \operatorname{atan2}(\omega, -10).$$

Note that the Matlab function atan2 is needed to correctly compute the arctangent. For details see "Which Quadrant Are We In?" In polar coordinates, we have

$$r(\omega) = \frac{10(1+\omega^2)^{1/2}}{\omega(1+\omega^2)} = \frac{10(1+\tan^2\theta_1(\omega))^{1/2}}{\tan\theta_1(\omega)(100+\tan^2\theta_1(\omega))^{1/2}}.$$

In addition, we see that

$$\begin{aligned} \operatorname{Re}(L(\jmath\omega)) &= x = r(\omega)\sin(\theta_1(\omega)-\theta_2(\omega)), \\ \operatorname{Im}(L(\jmath\omega)) &= y = -r(\omega)\cos(\theta_1(\omega)-\theta_2(\omega)), \end{aligned}$$

which implies that

$$\frac{x}{y} = -\tan(\theta_1(\omega)-\theta_2(\omega)).$$

The Nyquist plot is the "shifted strophoid" shown in Figure 14. The Cartesian equation is

$$\begin{aligned} &12100x^4 + 36300x^6y^2 - 53240x^4y^2 + 36300x^4y^4 + 12100x^2y^6 \\ &\quad - 43681x^2y^4 + 12100x^2y^2 - 14641y^6 + 12100x^8 - 24200x^6 = 0. \end{aligned} \tag{68}$$

■

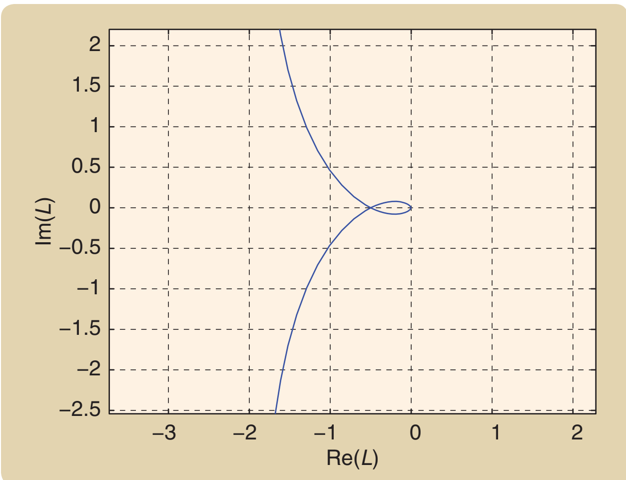

FIGURE 13 The Nyquist plot for the third-order Type I loop transfer function $L(s) = 1/[s(s+1)^2]$. This curve which is a shifted strophoid has an asymptote at -2. The polar equation is $r(\omega) = (1+\cos 2\theta(\omega))/(2\tan\theta(\omega))$, and the Cartesian equation is $4y^4x + 8x^3y^2 + 12x^2y^2 + 8y^4 + 4x^5 + 4x^4 + x^3 = 0$.

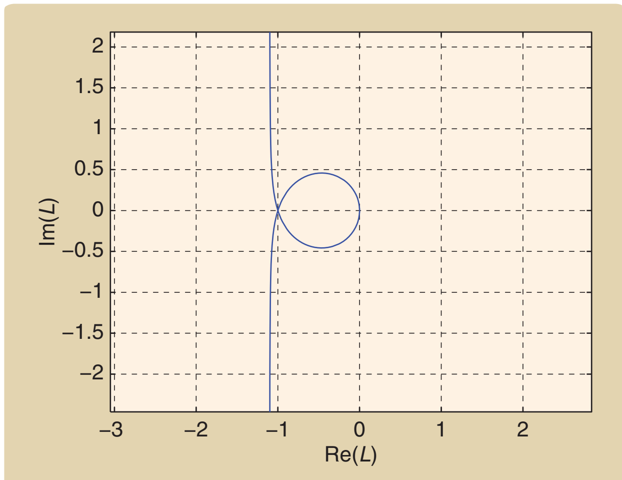

FIGURE 14 The Nyquist plot for the second-order loop transfer function $L(s) = (s+1)/[s(0.1s-1)]$. This curve, which is a "shifted strophoid," has an asymptote at -1.1. The polar equation is $r(\omega) = 10(1+\tan^2\theta_1(\omega))^{0.5}/[\tan\theta_1(\omega)(100+\tan^2\theta_1(\omega))^{0.5}]$, and the Cartesian equation is (68).

Which Quadrant Are We In?

The classic trigonometric function arctangent $\tan^{-1}$, referred to as atan in Matlab, may give the wrong answer for the phase of the complex quantity $z = x + \jmath y$. In particular, the Matlab computation phi $=$ atan(y/x) may give the wrong answer if the signs of the real and imaginary parts x and y are used to form the sign for the ratio y/x of the imaginary part and the real part. In this way, the information on the proper quadrant may be lost. Therefore, we must keep the signs of the real x and the imaginary y parts separate so that the correct quadrant can be identified to yield the right answer as seen in Figure S2. One of the jewels in Matlab is the four-quadrant arctangent function phi = atan2(y, x), $-\pi \le$ phi $\le \pi$. The function atan2 in Matlab identifies the correct quadrant by keeping track of the signs of x and y to yield the correct answer for the inverse tangent.

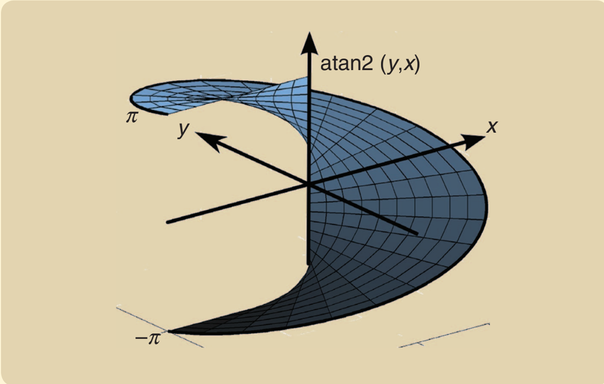

FIGURE S2 Illustration of the four-quadrant arctangent. The function atan2 in Matlab is needed to obtain the correct phase angle in arctangent computations.

Although the shape of the Nyquist curve resembles a shifted strophoid, this example does not satisfy the definition of a shifted strophoid.

Example 7: Strophoid

Consider the improper loop transfer function

$$L(s) = \frac{(s^2+1)(s+1)}{1-s^2}. \tag{69}$$

For $s = \jmath\omega$ and $\omega > 0$, we have

$$\begin{aligned} L(\jmath\omega) &= \frac{(1-\omega^2)(\jmath\omega+1)}{1+\omega^2} = \frac{(1-\omega^2)(1+\omega^2)^{1/2}}{1+\omega^2} e^{\jmath\theta(\omega)} \\ &= r(\omega)(\cos\theta(\omega) - \jmath\sin\theta(\omega)), \end{aligned}$$

where

$$r(\omega) = \frac{(1-\omega^2)(1+\omega^2)^{1/2}}{1+\omega^2} = \cos 2\theta(\omega)\sec\theta(\omega).$$

Moreover, we have

$$\frac{y}{x} = \tan\theta(\omega).$$

The Nyquist plot is the strophoid shown in Figure 15. The Cartesian equation is

$$y^2 = \frac{(1-x)x^2}{1+x}. \tag{70}$$

■

Example 8: Cayley's Sextic

Consider the loop transfer function

$$L(s) = \frac{1}{(s+1)^3}. \tag{71}$$

For $s = \jmath\omega$ and $\omega > 0$, we have

$$\begin{aligned} L(\jmath\omega) &= \frac{1}{(\jmath\omega+1)^3} = \frac{1}{(\omega^2+1)^{3/2}} e^{-\jmath 3\theta(\omega)} \\ &= r(\omega)(\cos 3\theta(\omega)) - \jmath\sin 3\theta(\omega)). \end{aligned}$$

The polar equation is

$$r(\omega) = \frac{1}{(\omega^2+1)^{3/2}} = \cos^3\theta(\omega) = \frac{1}{4}(3\cos\theta(\omega) + \cos 3\theta(\omega)).$$

Moreover, we have

$$\frac{y}{x} = -\tan 3\theta(\omega).$$

The Nyquist plot is the Cayley's sextic with $a = 1/4$ shown in Figure 16. The Cartesian equation is

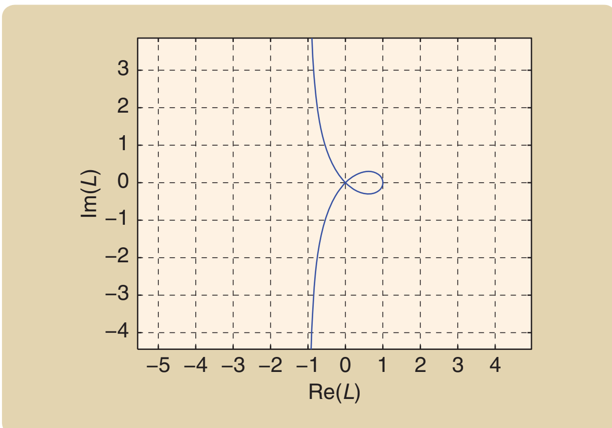

FIGURE 15 The Nyquist plot for the improper second-order loop transfer function $L(s) = (s^2+1)(s+1)/(1-s^2)$. This curve, which is a strophoid has an asymptote at -1. The polar equation is $r(\omega) = (\cos 2\theta(\omega))\sec\theta(\omega)$, and the Cartesian equation is $y^2 = x^2(1-x)/(1+x)$.

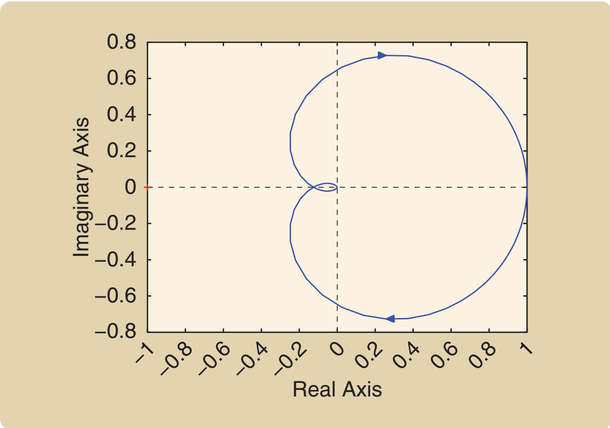

FIGURE 16 The Nyquist plot for the third-order loop transfer function $L(s) = 1/(s+1)^3$. This curve, which is a Cayley's sextic, has the polar equation $r(\omega) = 0.25(3\cos\theta(\omega) + r(\omega) = 0.25(3\cos\theta(\omega) + \cos 3\theta(\omega))$, and the Cartesian equation $4(x^2+y^2-0.25x)^3-(27/16)(x^2+y^2)^2 = 0$.

$$4\left(x^2 + y^2 - \frac{1}{4}x\right)^3 - \frac{27}{16}(x^2+y^2)^2 = 0. \qquad (72)$$

■

Example 9: Folium of Kepler

Consider the loop transfer function

$$L(s) = \frac{1}{(s-1)(s+1)^2}. \qquad (73)$$

For $s = \jmath\omega$ and $\omega > 0$, we have

$$\begin{aligned} L(\jmath\omega) &= \frac{1}{(\jmath\omega - 1)(\jmath\omega+1)^2} \\ &= \frac{1}{(1+\omega^2)^{3/2}} e^{-\jmath(\theta_1(\omega)+2\theta_2(\omega))} \\ &= r(\omega)(\cos(\theta_1(\omega)+2\theta_2(\omega)) - \jmath\sin(\theta_1(\omega)+2\theta_2(\omega))), \end{aligned}$$

where

$$\theta_1(\omega) = \operatorname{atan2}(\omega,-1), \quad \theta_2(\omega) = \operatorname{atan2}(\omega,1),$$

and

$$-\omega = \tan\theta_1(\omega), \quad \omega = \tan\theta_2(\omega).$$

Therefore, we have

$$\theta_1(\omega) = -\theta_2(\omega).$$

In polar coordinates we obtain

$$\begin{aligned} r(\omega) = \frac{1}{(1+\omega^2)^{3/2}} &= \frac{1}{(1+\tan^2\theta_1(\omega))^{3/2}} \\ &= \cos\theta_1(\omega)(\sin^2\theta_1(\omega) - 1), \end{aligned}$$

which implies that

$$\frac{y}{x} = -\tan\theta_2(\omega) = -\omega.$$

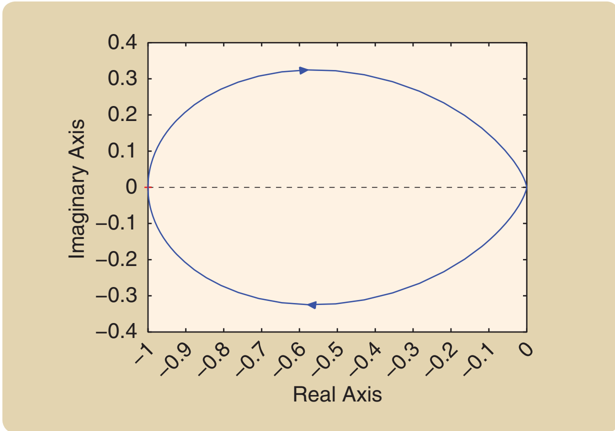

FIGURE 17 The Nyquist plot for the third-order loop transfer function $L(s) = 1/[(s-1)(s+1)^2]$. This curve, which is the folium of Kepler has the polar equation $r(\omega) = (\cos\theta_1(\omega))(\sin^2\theta_1(\omega) - 1)$, and the Cartesian equation $(x^2+y^2)[x(x-1)+y^2] + xy^2 = 0$.

The Nyquist plot is the folium of Kepler shown in Figure 17. The Cartesian equation is

$$(x^2+y^2)[x(x-1)+y^2] + xy^2 = 0. \qquad (74)$$

■

Example 10: Nephroid

Consider the exactly proper [13] loop transfer function

$$L(s) = \frac{2(s+1)(s^2-4s+1)}{(s-1)^3} = \frac{3(s+1)}{(s-1)} - \frac{(s+1)^3}{(s-1)^3}. \qquad (75)$$

For $s = \jmath\omega$ and $\omega > 0$,

$$\begin{aligned} L(\jmath\omega) &= \frac{3(\jmath\omega+1)}{(\jmath\omega-1)} - \frac{(\jmath\omega+1)^3}{(\jmath\omega-1)^3} \\ &= 3e^{\jmath(\theta_1(\omega)-\pi+\theta_2(\omega))} - e^{\jmath(3\theta_1(\omega)-\pi+3\theta_2(\omega))} \\ &= 3(-\cos 2\theta_1(\omega) - \jmath\sin 2\theta_1(\omega)) \\ &\quad - (-\cos 6\theta_1(\omega) - \jmath\sin 6\theta_1(\omega)), \end{aligned}$$

where

$$\begin{aligned} \theta_1(\omega) &= \tan^{-1}(\omega), \ \omega = \tan\theta_1(\omega), \\ \theta_2(\omega) &= \tan^{-1}(\omega), \ \omega = \tan\theta_2(\omega). \end{aligned}$$

Furthermore, we see that

$$\begin{aligned} \operatorname{Re}(L(\jmath\omega)) &= x = -3\cos 2\theta_1(\omega) + \cos 6\theta_1(\omega), \\ \operatorname{Im}(L(\jmath\omega)) &= y = -3\sin 2\theta_1(\omega) + \sin 6\theta_1(\omega), \end{aligned}$$

which leads to the relation

$$\frac{x}{y} = \frac{-3\cos 2\theta_1(\omega) + \cos 6\theta_1(\omega)}{-3\sin 2\theta_1(\omega) + \sin 6\theta_1(\omega)}.$$

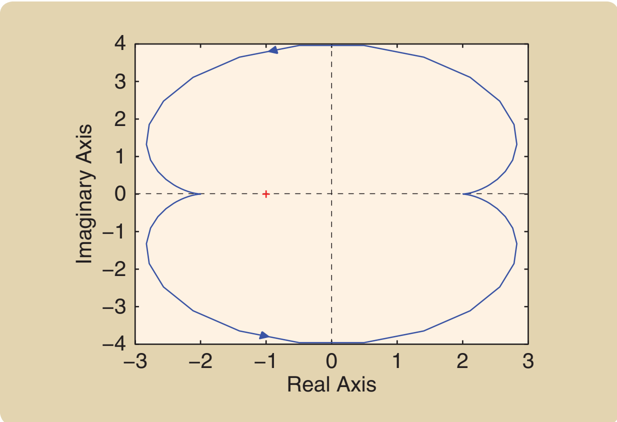

FIGURE 18 The Nyquist plot for the third-order loop transfer function $L(s) = 2(s+1)(s^2-4s+1)/(s-1)^3$. This curve, which is a nephroid has the Cartesian equation $(x^2+y^2-4)^3-108y^2=0$.

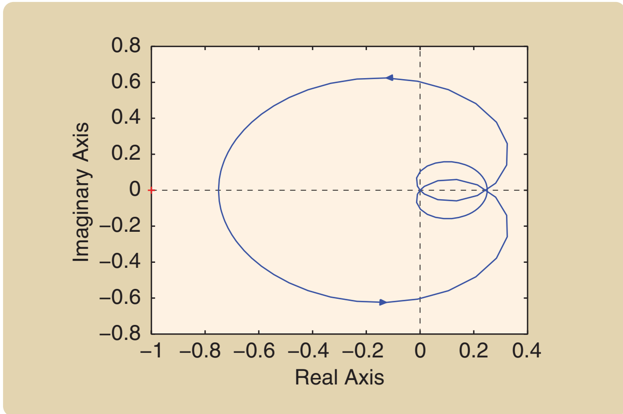

FIGURE 19 The Nyquist plot of the third-order loop transfer function with a pair of zeros on the $\jmath\omega$ axis $L(s) = (s+1)(s^2+3)/[4(s-1)^3]$. This curve, which is a nephroid of Freeth has the Cartesian equation given by (78).

The Nyquist plot is the nephroid shown in Figure 18. The Cartesian equation is

$$(x^2+y^2-4)^3 - 108y^2 = 0. \tag{76}$$

■

Example 11: Nephroid of Freeth

Consider the exactly proper loop transfer function

$$L(s) = \frac{(s+1)(s^2+3)}{4(s-1)^3}. \tag{77}$$

For $s = \jmath\omega$ we have

$$\begin{aligned} L(\jmath\omega) &= \frac{(\jmath\omega+1)(-\omega^2+3)}{4(\jmath\omega-1)^3} \\ &= \begin{cases} r(\omega)(-\cos 4\theta(\omega) - \jmath\sin 4\theta(\omega)), & 0<\omega<\sqrt{3}, \\ -r(\omega)(\cos 4\theta(\omega) + \jmath\sin 4\theta(\omega)), & \omega > \sqrt{3}, \end{cases} \end{aligned}$$

where

$$\begin{aligned} r(\omega) &= \frac{(1+\omega^2)^{1/2}(-\omega^2+3)}{4(1+\omega^2)^{3/2}} = \frac{(-\omega^2+3)}{4(1+\omega^2)}, \\ \theta(\omega) &= \tan^{-1}(\omega),\ \omega = \tan\theta(\omega). \end{aligned}$$

Furthermore, we obtain the relations

$$\begin{aligned} \mathrm{Re}(L(\jmath\omega)) = x &= \begin{cases} -r(\omega)\cos 4\theta(\omega), & 0<\omega<\sqrt{3}, \\ -r(\omega)\cos 4\theta(\omega), & \omega>\sqrt{3}, \end{cases} \\ \mathrm{Im}(L(\jmath\omega)) = y &= \begin{cases} -r(\omega)\sin 4\theta(\omega), & 0<\omega<\sqrt{3}, \\ -r(\omega)\sin 4\theta(\omega), & \omega>\sqrt{3}, \end{cases} \end{aligned}$$

which lead to

$$\frac{y}{x} = \tan 4\theta(\omega).$$

The Nyquist plot is the nephroid of Freeth with $a = 1/4$ shown in Figure 19. The Cartesian equation is

$$(x^2+y^2)\left(x^2+y^2-\frac{1}{16}\right)^2 - \frac{1}{4}\left(x^2+y^2-\frac{1}{4}x\right)^2 = 0. \tag{78}$$

■

Example 12: Shifted Nephroid of Freeth

Consider the strictly proper loop transfer function

$$L(s) = \frac{s^2+1}{(s-1)^3}. \tag{79}$$

For $s = \jmath\omega$, we have

$$\begin{aligned} L(\jmath\omega) &= \frac{-\omega^2+1}{(\jmath\omega-1)^3} \\ &= \begin{cases} -r(\omega)(\cos 3\theta(\omega) + \jmath\sin 3\theta(\omega)), & 0<\omega<1, \\ -r(\omega)(\cos 3\theta(\omega) + \jmath\sin 3\theta(\omega)), & \omega>1. \end{cases} \end{aligned}$$

In polar coordinates we have

$$r(\omega) = \frac{-\omega^2+1}{(1+\omega^2)^{3/2}} = \cos\theta(\omega)\cos 2\theta(\omega).$$

Moreover, we observe that

$$\begin{aligned} \mathrm{Re}(L(\jmath\omega)) = x &= \begin{cases} -r(\omega)\cos 3\theta(\omega), & 0<\omega<1, \\ -r(\omega)\cos 3\theta(\omega), & \omega>1, \end{cases} \\ \mathrm{Im}(L(\jmath\omega)) = y &= \begin{cases} -r(\omega)\sin 3\theta(\omega), & 0<\omega<1, \\ -r(\omega)\sin 3\theta(\omega), & \omega>1, \end{cases} \end{aligned}$$

which implies that

$$\frac{y}{x} = \tan 3\theta(\omega).$$

The Nyquist plot is the shifted nephroid of Freeth shown in Figure 20. The Cartesian equation is

$$\left(\left(x+\frac{1}{4}\right)^2+y^2\right)\left(\left(x+\frac{1}{4}\right)^2+y^2-\frac{1}{16}\right)^2$$
$$-\frac{1}{4}\left(\left(x+\frac{1}{4}\right)^2+y^2-\frac{1}{4}\left(x+\frac{1}{4}\right)\right)^2=0. \quad (80)$$

■

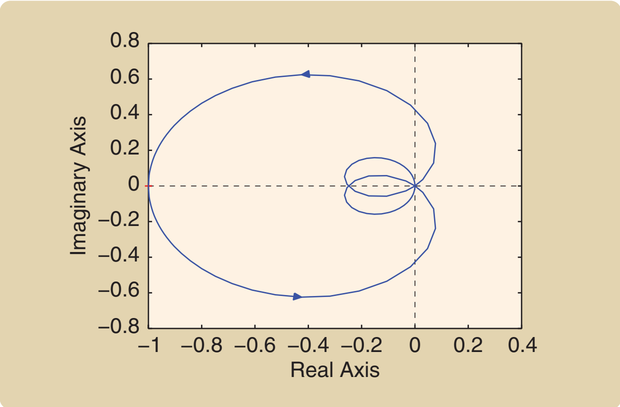

FIGURE 20 The Nyquist plot for the third-order loop transfer function with a pair of zeros on the $j\omega$ axis $L(s)=(s^2+1)/(s-1)^3$. This curve, which is a shifted nephroid of Freeth has the Cartesian equation given by (80).

Root Locus and the Plane Curves

Plane curves also appear in the root locus problems [12]. For example, the root locus associated with [2, Ex. 5.13, p. 255] with the loop transfer function

$$L(s)=\frac{s+1}{s^2(s+9)}$$

is the *trisectrix* of Maclaurin [7] shown in Figure S3. The curve was first studied by the Scottish mathematician C. Maclaurin in 1742. The root locus associated with [2, Ex. 6.11, p. 255] resembles a limaçon.

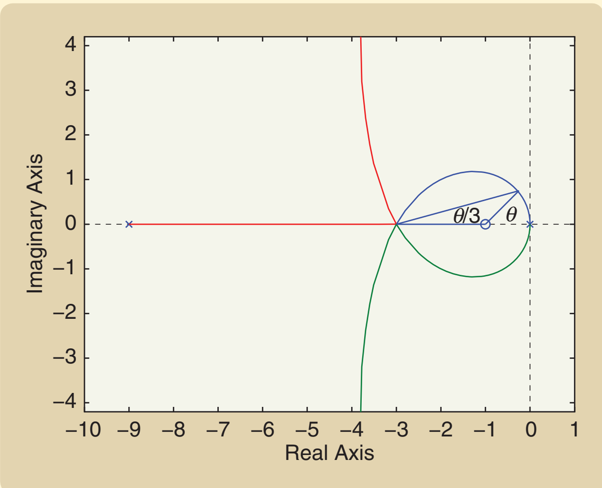

FIGURE S3 Illustration of the root locus shaped as a trisectrix. This root locus is the plane curve known as the trisectrix of Maclaurin.

CONCLUSIONS

We have shown that the shapes of many Nyquist plots are identical to familiar and well-studied plane curves. This observation can provide additional insight into the shapes of Nyquist plots. Knowing the shape of the Nyquist plot can also provide additional useful information about the system beyond stability. Table 1 shows a summary of the examples and the corresponding shapes of the Nyquist plots. Some plane curves, such as the folium of Descartes [7], are not symmetric with respect to the horizontal axis, and thus are not related to Nyquist curves. Plane curves also appear in root locus problems. For details, see "Root Locus and the Plane Curves." Many other examples of the Nyquist plots that are related to plane curves can be constructed using the techniques discussed here.

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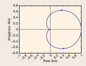
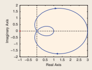
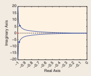
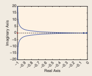
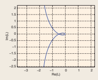
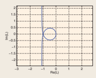
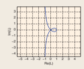
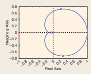
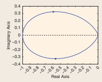
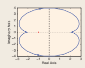
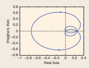
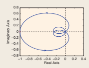
[1] H. Nyquist, "Regeneration theory," *Bell Syst. Tech. J.*, vol. 11, pp. 126–147, Jan. 1932.

[2] G. F. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, 5th ed. Englewood Cliffs, NJ: Prentice-Hall, 2006.

TABLE 1 Summary of the example loop transfer functions and the associated plane curves. These examples illustrate that the shapes of the Nyquist plots are identical to the well-studied plane curves. The Nyquist curves can be described in either polar coordinates or Cartesian coordinates.

Loop Transfer Function	Plane Curve	Nyquist Plot
$L(s) = \dfrac{1}{(s+1)^2}$	Cardioid	
$L(s) = \dfrac{s^2+3}{(s+1)^2}$	Limaçon	
$L(s) = \dfrac{1}{s(s+1)}$	Cissoid of Diocles	
$L(s) = \dfrac{s^2}{s+1}$	Cissoid of Diocles for an improper system	
$L(s) = \dfrac{1}{s(s+1)^2}$	Shifted strophoid	
$L(s) = \dfrac{s+1}{s(0.1s-1)}$	"Shifted strophoid"	
$L(s) = \dfrac{(s^2+1)(s+1)}{1-s^2}$	Strophoid	
$L(s) = \dfrac{1}{(s+1)^3}$	Cayley's sextic	
$L(s) = \dfrac{1}{(s-1)(s+1)^2}$	Folium of Kepler	
$L(s) = \dfrac{2(s+1)(s^2-4s+1)}{(s-1)^3}$	Nephroid	
$L(s) = \dfrac{(s+1)(s^2+3)}{4(s-1)^3}$	Nephroid of Freeth	
$L(s) = \dfrac{s^2+1}{(s-1)^3}$	Shifted nephroid of Freeth	

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