

Appendix WD

Ackermann's Formula for Pole Placement

Given the plant and state-variable equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad (\text{WD.1})$$

our objective is to find a state-feedback control law

$$u = -\mathbf{K}\mathbf{x}, \quad (\text{WD.2})$$

such that the closed-loop characteristic polynomial is

$$\alpha_c(s) = \det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}). \quad (\text{WD.3})$$

First we have to select $\alpha_c(s)$, which determines where the poles are to be shifted; then we have to find \mathbf{K} such that Eq. (WD.3) will be satisfied. Our technique is based on transforming the plant equation into control canonical form.

We begin by considering the effect of an arbitrary nonsingular transformation of the state, as

$$\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}, \quad (\text{WD.4})$$

where $\bar{\mathbf{x}}$ is the new transformed state. The state equations in the new coordinates, from Eq. (WD.4), are

$$\dot{\mathbf{x}} = \mathbf{T}\dot{\bar{\mathbf{x}}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \mathbf{A}\mathbf{T}\bar{\mathbf{x}} + \mathbf{B}u, \quad (\text{WD.5})$$

$$\dot{\bar{\mathbf{x}}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\bar{\mathbf{x}} + \mathbf{T}^{-1}\mathbf{B}u = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}u. \quad (\text{WD.6})$$

Now the controllability matrix for the original state,

$$\mathbf{C}_x = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}], \quad (\text{WD.7})$$

provides a useful transformation matrix. We can also define the controllability matrix for the transformed state:

$$\mathbf{C}_{\bar{x}} = [\bar{\mathbf{B}} \quad \bar{\mathbf{A}}\bar{\mathbf{B}} \quad \bar{\mathbf{A}}^2\bar{\mathbf{B}} \quad \dots \quad \bar{\mathbf{A}}^{n-1}\bar{\mathbf{B}}]. \quad (\text{WD.8})$$

The two controllability matrices are related by

$$\mathbf{C}_{\bar{x}} = [\mathbf{T}^{-1}\mathbf{B} \quad \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{T}^{-1}\mathbf{B} \quad \dots] = \mathbf{T}^{-1}\mathbf{C}_x \quad (\text{WD.9})$$

and the transformation matrix

$$\mathbf{T} = \mathbf{C}_x \mathbf{C}_{\bar{x}}^{-1}. \quad (\text{WD.10})$$

From Eqs. (WD.9) and (WD.10), we can draw some important conclusions. From Eq. (WD.9), we see if \mathbf{C}_x is nonsingular, then for any nonsingular \mathbf{T} , $\mathbf{C}_{\bar{x}}$ is also nonsingular. This means that a similarity

transformation of the state does not change the controllability properties of a system. We can look at this in another way. Suppose we would like to find a transformation to take the system (\mathbf{A}, \mathbf{B}) into control canonical form. As we shall shortly see, $\mathcal{C}_{\bar{\mathbf{x}}}$ in that case is *always* nonsingular. From Eq. (WD.9), we see a nonsingular \mathbf{T} will always exist if and only if $\mathcal{C}_{\mathbf{x}}$ is nonsingular. We derive the following theorem.

Theorem WD.1 *We can always transform (\mathbf{A}, \mathbf{B}) into control canonical form if and only if the system is controllable.*

Let us take a closer look at the control canonical form and treat the third-order case, although the results are true for any n th-order case:

$$\bar{\mathbf{A}} = \mathbf{A}_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{\mathbf{B}} = \mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad (\text{WD.11})$$

The controllability matrix, by direct computation, is

$$\mathcal{C}_{\bar{\mathbf{x}}} = \mathcal{C}_c = \begin{bmatrix} 1 & -a_1 & a_1^2 - a_2 \\ 0 & 1 & -a_1 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{WD.12})$$

Because this matrix is upper triangular with ones along the diagonal, it is always invertible. Also note the last row of $\mathcal{C}_{\bar{\mathbf{x}}}$ is the unit vector with all zeros, except for the last element, which is unity. We shall use this fact in the following.

As we pointed out in Section 7.5, the design of a control law for the state $\bar{\mathbf{x}}$ is trivial if the state equations happen to be in control canonical form. The characteristic equation is

$$s^3 + a_1 s^2 + a_2 s + a_3 = 0, \quad (\text{WD.13})$$

and the characteristic equation for the closed-loop system comes from

$$\mathbf{A}_{cl} = \mathbf{A}_c - \mathbf{B}_c \mathbf{K}_c \quad (\text{WD.14})$$

and has the coefficients shown:

$$s^3 + (a_1 + K_{c1})s^2 + (a_2 + K_{c2})s + (a_3 + K_{c3}) = 0. \quad (\text{WD.15})$$

To obtain the desired closed-loop pole locations, we must make the coefficients of s in Eq. (WD.15) match those in

$$\alpha_c(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3, \quad (\text{WD.16})$$

so

$$a_1 + K_{c1} = \alpha_1, \quad a_2 + K_{c2} = \alpha_2, \quad a_3 + K_{c3} = \alpha_3, \quad (\text{WD.17})$$

or in vector form,

$$\mathbf{a} + \mathbf{K}_c = \boldsymbol{\alpha}, \quad (\text{WD.18})$$

where \mathbf{a} and $\boldsymbol{\alpha}$ are row vectors containing the coefficients of the characteristic polynomials of the open-loop and closed-loop systems, respectively.

We now need to find a relationship between these polynomial coefficients and the matrix \mathbf{A} . The requirement is achieved by the Cayley–Hamilton theorem, which states that a matrix satisfies its own characteristic polynomial. For \mathbf{A}_c , this means that

$$\mathbf{A}_c^n + a_1 \mathbf{A}_c^{n-1} + a_2 \mathbf{A}_c^{n-2} + \cdots + a_n \mathbf{I} = \mathbf{0}. \quad (\text{WD.19})$$

Now suppose we form the polynomial $\alpha_c(\mathbf{A})$, which is the *closed-loop* characteristic polynomial with the matrix \mathbf{A} substituted for the complex variable s :

$$\alpha_c(\mathbf{A}_c) = \mathbf{A}_c^n + \alpha_1 \mathbf{A}_c^{n-1} + \alpha_2 \mathbf{A}_c^{n-2} + \cdots + \alpha_n \mathbf{I}. \quad (\text{WD.20})$$

If we solve Eq. (WD.19) for \mathbf{A}_c^n and substitute into Eq. (WD.20), we find that

$$\alpha_c(\mathbf{A}_c) = (-a_1 + \alpha_1) \mathbf{A}_c^{n-1} + (-a_2 + \alpha_2) \mathbf{A}_c^{n-2} + \cdots + (-\alpha_n + \alpha_n) \mathbf{I}. \quad (\text{WD.21})$$

But, because \mathbf{A}_c has such a special structure, we observe that if we multiply it by the transpose of the n th unit vector, $\mathbf{e}_n^T = [0 \ \cdots \ 0 \ 1 \ 0]$, we get

$$\mathbf{e}_n^T \mathbf{A}_c = [0 \ \cdots \ 0 \ 1 \ 0] = \mathbf{e}_{n-1}^T, \quad (\text{WD.22})$$

as we can see from Eq. (WD.11). If we multiply this vector again by \mathbf{A}_c , getting

$$\begin{aligned} (\mathbf{e}_n^T \mathbf{A}_c) \mathbf{A}_c &= [0 \ \cdots \ 0 \ 1 \ 0] \mathbf{A}_c \\ &= [0 \ \cdots \ 0 \ 1 \ 0 \ 0] = \mathbf{e}_{n-2}^T, \end{aligned} \quad (\text{WD.23})$$

and continue the process, successive unit vectors are generated until

$$\mathbf{e}_n^T \mathbf{A}_c^{n-1} = [1 \ 0 \ \cdots \ 0] = \mathbf{e}_1^T. \quad (\text{WD.24})$$

Therefore, if we multiply Eq. (WD.21) by \mathbf{e}_n^T , we find that

$$\begin{aligned} \mathbf{e}_n^T \alpha_c(\mathbf{A}_c) &= (-a_1 + \alpha_1) \mathbf{e}_1^T + (-a_2 + \alpha_2) \mathbf{e}_2^T + \cdots + (-a_n + \alpha_n) \mathbf{e}_n^T \\ &= [K_{c1} \ K_{c2} \ \cdots \ K_{cn}] = \mathbf{K}_c, \end{aligned} \quad (\text{WD.25})$$

where we use Eq. (WD.18), which relates \mathbf{K}_c to \mathbf{a} and α .

We now have a compact expression for the gains of the system in control canonical form as represented in Eq. (WD.25). However, we still need the expression for \mathbf{K} , which is the gain on the original state. If $u = -\mathbf{K}_c \bar{\mathbf{x}}$, then $u = -\mathbf{K}_c \mathbf{T}^{-1} \mathbf{x}$, so

$$\begin{aligned} \mathbf{K} &= \mathbf{K}_c \mathbf{T}^{-1} = \mathbf{e}_n^T \alpha_c(\mathbf{A}_c) \mathbf{T}^{-1} \\ &= \mathbf{e}_n^T \alpha_c(\mathbf{T}^{-1} \mathbf{A} \mathbf{T}) \mathbf{T}^{-1} \\ &= \mathbf{e}_n^T \mathbf{T}^{-1} \alpha_c(\mathbf{A}). \end{aligned} \quad (\text{WD.26})$$

In the last step of Eq. (WD.26), we used the fact that $(\mathbf{T}^{-1} \mathbf{A} \mathbf{T})^k = \mathbf{T}^{-1} \mathbf{A}^k \mathbf{T}$ and that α_c is a polynomial, that is, a sum of the powers of \mathbf{A}_c . From Eq. (WD.9), we see that

$$\mathbf{T}^{-1} = \mathcal{C}_c \mathcal{C}_x^{-1}. \quad (\text{WD.27})$$

With this substitution, Eq. (WD.26) becomes

$$\mathbf{K} = \mathbf{e}_n^T \mathcal{C}_c \mathcal{C}_x^{-1} \alpha_c(\mathbf{A}). \quad (\text{WD.28})$$

Ackermann's formula

Now, we use the observation made earlier for Eq. (WD.12) that the last row of \mathcal{C}_c , which is $\mathbf{e}_n^T \mathcal{C}_c$, is again \mathbf{e}_n^T . We finally obtain Ackermann's formula:

$$\mathbf{K} = \mathbf{e}_n^T \mathcal{C}_x^{-1} \alpha_c(\mathbf{A}). \quad (\text{WD.29})$$

We note again that forming the explicit inverse of \mathcal{C}_x is not advisable for numerical accuracy. Thus we need to solve \mathbf{b}^T such that

$$\mathbf{e}_n^T \mathcal{C}_x^{-1} = \mathbf{b}^T. \quad (\text{WD.30})$$

We solve the linear set of equations

$$\mathbf{b}^T \mathcal{C}_x = \mathbf{e}_n^T, \quad (\text{WD.31})$$

then compute

$$\mathbf{K} = \mathbf{b}^T \alpha_c(\mathbf{A}). \quad (\text{WD.32})$$

Ackermann's formula, Eq. (WD.29), even though elegant, is not recommended for systems with a large number of state variables. Even if it is used, Eqs. (WD.31) and (WD.32) are recommended for better numerical accuracy.