

Appendix WA

A Review of Complex Variables

This appendix is a brief summary of some results on complex variables theory, with emphasis on the facts needed in control theory. For a comprehensive study of basic complex variables theory, see standard textbooks such as Brown and Churchill (1996) or Marsden and Hoffman (1998).

WA.1 Definition of a Complex Number

The complex numbers are distinguished from purely real numbers in that they also contain the **imaginary operator**, which we shall denote as j . By definition,

$$j^2 = -1 \quad \text{or} \quad j = \sqrt{-1}. \quad (\text{WA.1})$$

A **complex number** may be defined as

$$A = \sigma + j\omega, \quad (\text{WA.2})$$

where σ is the real part and ω is the imaginary part, denoted, respectively, as

$$\sigma = \text{Re}(A), \quad \omega = \text{Im}(A). \quad (\text{WA.3})$$

Note the imaginary part of A is itself a real number.

Graphically, we may represent the complex number A in two ways. In the Cartesian coordinate system (see Fig. WA.1a), A is represented by a single point in the complex plane. In the polar coordinate system, A is represented by a vector with length r and an angle θ ; the angle is measured in radians counter-clockwise from the positive real axis (see Fig. WA.1b). In polar form, the complex number A is denoted by

$$A = |A| \cdot \angle \arg A = r \cdot \angle \theta = re^{j\theta}, \quad 0 \leq \theta \leq 2\pi, \quad (\text{WA.4})$$

where r —called the **magnitude**, **modulus**, or **absolute value** of A —is the length of the vector representing A , namely,

$$r = |A| = \sqrt{\sigma^2 + \omega^2}, \quad (\text{WA.5})$$

and where θ is given by

$$\tan \theta = \frac{\omega}{\sigma} \quad (\text{WA.6})$$

or

$$\theta = \arg(A) = \tan^{-1} \left(\frac{\omega}{\sigma} \right). \quad (\text{WA.7})$$

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Figure WA.1

The complex number A represented in
(a) Cartesian and
(b) polar coordinates

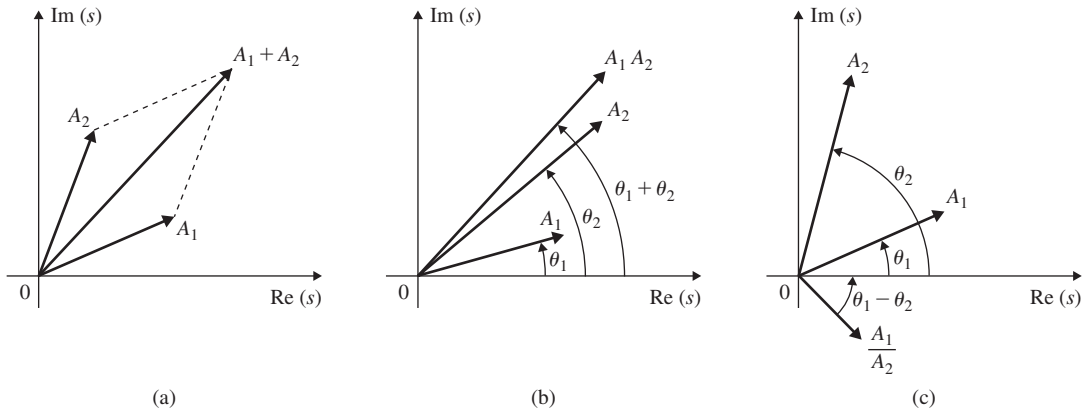
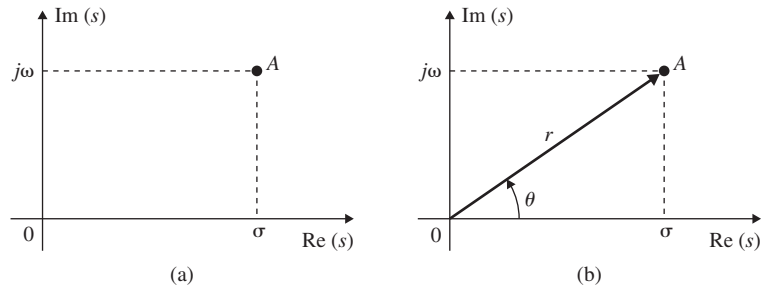


Figure WA.2

Arithmetic of complex numbers: (a) addition; (b) multiplication; (c) division

Care must be taken to compute the correct value of the angle, depending on the sign of the real and imaginary parts (that is, one must find the quadrant in which the complex number lies).

The **conjugate** of A is defined as

$$A^* = \sigma - j\omega. \quad (\text{WA.8})$$

Therefore,

$$(A^*)^* = A, \quad (\text{WA.9})$$

$$(A_1 \pm A_2)^* = A_1^* \pm A_2^*, \quad (\text{WA.10})$$

$$\left(\frac{A_1}{A_2}\right)^* = \frac{A_1^*}{A_2^*}, \quad (\text{WA.11})$$

$$(A_1 A_2)^* = A_1^* A_2^*, \quad (\text{WA.12})$$

$$\text{Re}(A) = \frac{A + A^*}{2}, \quad \text{Im}(A) = \frac{A - A^*}{2j}, \quad (\text{WA.13})$$

$$A A^* = (|A|)^2. \quad (\text{WA.14})$$

WA.2 Algebraic Manipulations

WA.2.1 Complex Addition

If we let

$$A_1 = \sigma_1 + j\omega_1 \quad \text{and} \quad A_2 = \sigma_2 + j\omega_2, \quad (\text{WA.15})$$

then

$$A_1 + A_2 = (\sigma_1 + j\omega_1) + (\sigma_2 + j\omega_2) = (\sigma_1 + \sigma_2) + j(\omega_1 + \omega_2). \quad (\text{WA.16})$$

Because each complex number is represented by a vector extending from the origin, we can add or subtract complex numbers graphically. The sum is obtained by adding the two vectors. This we do by constructing a parallelogram and finding its diagonal, as shown in Fig. WA.2a. Alternatively, we could start at the tail of one vector, draw a vector parallel to the other vector, then connect the origin to the new arrowhead.

Complex subtraction is very similar to complex addition.

WA.2.2 Complex Multiplication

For two complex numbers defined according to Eq. (WA.15),

$$\begin{aligned} A_1 A_2 &= (\sigma_1 + j\omega_1)(\sigma_2 + j\omega_2) \\ &= (\sigma_1\sigma_2 - \omega_1\omega_2) + j(\omega_1\sigma_2 + \sigma_1\omega_2). \end{aligned} \quad (\text{WA.17})$$

The product of two complex numbers may be obtained graphically using polar representations, as shown in Fig. WA.2b.

WA.2.3 Complex Division

The division of two complex numbers is carried out by **rationalization**. This means that both the numerator and denominator in the ratio are multiplied by the conjugate of the denominator:

$$\begin{aligned} \frac{A_1}{A_2} &= \frac{A_1 A_2^*}{A_2 A_2^*} \\ &= \frac{(\sigma_1\sigma_2 + \omega_1\omega_2) + j(\omega_1\sigma_2 - \sigma_1\omega_2)}{\sigma_2^2 + \omega_2^2}. \end{aligned} \quad (\text{WA.18})$$

From Eq. (WA.4), it follows that

$$A^{-1} = \frac{1}{r} e^{-j\theta}, \quad r \neq 0. \quad (\text{WA.19})$$

Also, if $A_1 = r_1 e^{j\theta_1}$ and $A_2 = r_2 e^{j\theta_2}$, then

$$A_1 A_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}, \quad (\text{WA.20})$$

where $|A_1 A_2| = r_1 r_2$ and $\arg(A_1 A_2) = \theta_1 + \theta_2$, and

$$\frac{A_1}{A_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}, \quad r_2 \neq 0, \quad (\text{WA.21})$$

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where $\left| \frac{A_1}{A_2} \right| = \frac{r_1}{r_2}$ and $\arg\left(\frac{A_1}{A_2}\right) = \theta_1 - \theta_2$. The division of complex numbers may be carried out graphically in polar coordinates as shown in Fig. WA.2c.

EXAMPLE WA.1

Frequency Response of First-Order System

Find the magnitude and phase of the transfer function $G(s) = \frac{1}{s+1}$, where $s = j\omega$.

Solution. Substituting $s = j\omega$ and rationalizing, we obtain

$$\begin{aligned} G(j\omega) &= \frac{1}{\sigma+1+j\omega} \frac{\sigma+1-j\omega}{\sigma+1-j\omega} \\ &= \frac{\sigma+1-j\omega}{(\sigma+1)^2 + \omega^2}. \end{aligned}$$

Therefore, the magnitude and phase are

$$\begin{aligned} |G(j\omega)| &= \frac{\sqrt{(\sigma+1)^2 + \omega^2}}{(\sigma+1)^2 + \omega^2} = \frac{1}{\sqrt{(\sigma+1)^2 + \omega^2}}, \\ \arg(G(j\omega)) &= \tan^{-1} \left(\frac{\text{Im}(G(j\omega))}{\text{Re}(G(j\omega))} \right) = \tan^{-1} \left(\frac{-\omega}{\sigma+1} \right). \end{aligned}$$

WA.3 Graphical Evaluation of Magnitude and Phase

Consider the transfer function

$$G(s) = \frac{\prod_{i=1}^m (s + z_i)}{\prod_{i=1}^n (s + p_i)}. \quad (\text{WA.22})$$

The value of the transfer function for sinusoidal inputs is found by replacing s with $j\omega$. The gain and phase are given by $G(j\omega)$ and may be determined analytically or by a graphical procedure. Consider the pole-zero configuration for such a $G(s)$ and a point $s_0 = j\omega_0$ on the imaginary axis, as shown in Fig. WA.3. Also consider the vectors drawn from the poles and the zero to s_0 . The magnitude of the transfer function evaluated at $s_0 = j\omega_0$ is simply the ratio of the distance from the zero to the product of all the distances from the poles:

$$|G(j\omega_0)| = \frac{r_1}{r_2 r_3 r_4}. \quad (\text{WA.23})$$

The phase is given by the sum of the angles from the zero, minus the sum of the angles from the poles:

$$\arg G(j\omega_0) = \angle G(j\omega_0) = \theta_1 - (\theta_2 + \theta_3 + \theta_4). \quad (\text{WA.24})$$

This may be explained as follows: The term $s + z_1$ is a vector addition of its two components. We may determine this equivalently as $s - (-z_1)$, which amounts to translation of the vector $s + z_1$ starting at $-z_1$, as

Figure WA.3

Graphical
determination of
magnitude and phase

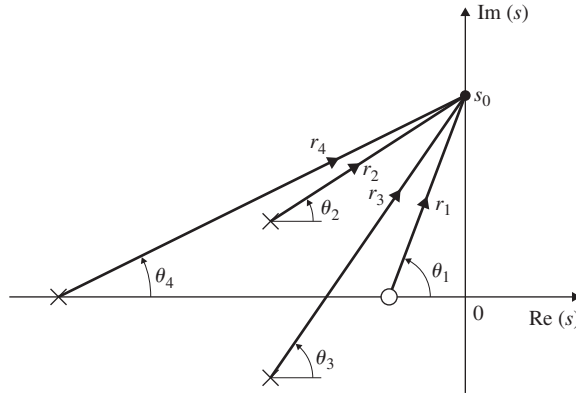
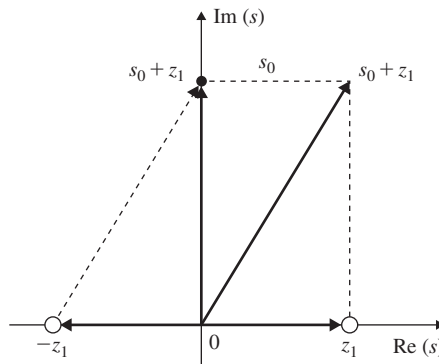

Figure WA.4

Illustration of graphical
computation of $s + z_1$



shown in Fig. WA.4. This means that a vector drawn from the zero location to s_0 is equivalent to $s + z_1$. The same reasoning applies to the poles. We reflect p_1 , p_2 , and p_3 about the origin to obtain the pole locations. Then the vectors drawn from $-p_1$, $-p_2$, and $-p_3$ to s_0 are the same as the vectors in the denominator represented in polar coordinates. Note that this method may also be used to evaluate s_0 at places in the complex plane besides the imaginary axis.

WA.4 Differentiation and Integration

The usual rules apply to complex differentiation. Let $G(s)$ be differentiable with respect to s . Then the derivative at s_0 is defined as

$$G'(s_0) = \lim_{s \rightarrow s_0} \frac{G(s) - G(s_0)}{s - s_0}, \quad (\text{WA.25})$$

provided that the limit exists. For conditions on the existence of the derivative, see Brown and Churchill (1996).

The standard rules also apply to integration, except that the constant of integration c is a complex constant:

$$\int G(s) ds = \int \text{Re}[G(s)] ds + j \int \text{Im}[G(s)] ds + c. \quad (\text{WA.26})$$

WA.5 Euler's Relations

Let us now derive an important relationship involving the complex exponential. If we define

$$A = \cos \theta + j \sin \theta, \quad (\text{WA.27})$$

where θ is in radians, then

$$\begin{aligned} \frac{dA}{d\theta} &= -\sin \theta + j \cos \theta = j^2 \sin \theta + j \cos \theta \\ &= j(\cos \theta + j \sin \theta) = jA. \end{aligned} \quad (\text{WA.28})$$

We collect the terms involving A to obtain

$$\frac{dA}{A} = j d\theta. \quad (\text{WA.29})$$

Integrating both sides of Eq. (WA.29) yields

$$\ln A = j\theta + c, \quad (\text{WA.30})$$

where c is a constant of integration. If we let $\theta = 0$ in Eq. (WA.30), we find that $c = 0$ or

$$A = e^{j\theta} = \cos \theta + j \sin \theta. \quad (\text{WA.31})$$

Similarly,

$$A^* = e^{-j\theta} = \cos \theta - j \sin \theta. \quad (\text{WA.32})$$

From Eqs. (WA.31) and (WA.32), it follows that

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad (\text{WA.33})$$

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}. \quad (\text{WA.34})$$

Euler's relations

WA.6 Analytic Functions

Let us assume G is a complex-valued function defined in the complex plane. Let s_0 be in the domain of G , which is assumed to be finite within some disk centered at s_0 . Thus, $G(s)$ is defined not only at s_0 but also at all points in the disk centered at s_0 . The function G is said to be **analytic** if its derivative exists at s_0 and at each point in the neighborhood of s_0 .

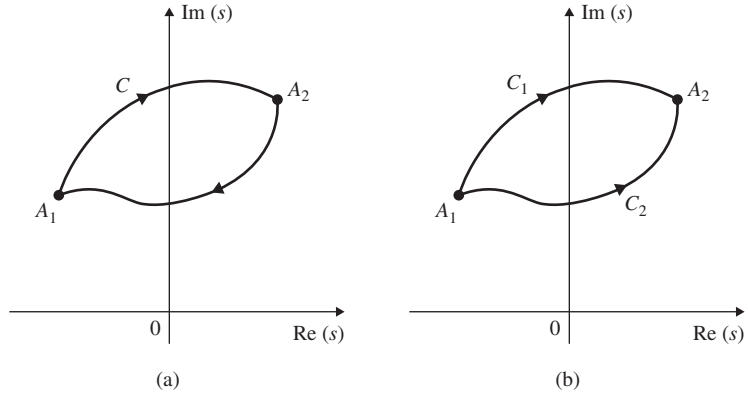
WA.7 Cauchy's Theorem

A **contour** is a piecewise-smooth arc that consists of a number of smooth arcs joined together. A **simple closed contour** is a contour that does not intersect itself and ends on itself. Let C be a closed contour as shown in Fig. WA.5a, and let G be analytic inside and on C . Cauchy's theorem states that

$$\oint_C G(s) ds = 0. \quad (\text{WA.35})$$

Figure WA.5

Contours in the s -plane:
 (a) a closed contour;
 (b) two different paths
 between A_1 and A_2



There is a corollary to this theorem: Let C_1 and C_2 be two paths connecting the points A_1 and A_2 as in Fig. WA.5b. Then,

$$\int_{C_1} G(s)ds = \int_{C_2} G(s)ds. \quad (\text{WA.36})$$

WA.8 Singularities and Residues

If a function $G(s)$ is not analytic at s_0 , but is analytic at some point in every neighborhood of s_0 , it is said to be a **singularity**. A singular point is said to be an **isolated singularity** if $G(s)$ is analytic everywhere else in the neighborhood of s_0 except at s_0 . Let $G(s)$ be a **rational function** (that is, a ratio of polynomials). If the numerator and denominator are both analytic, then $G(s)$ will be analytic except at the locations of the poles (that is, at the roots of the denominator). All singularities of rational algebraic functions are pole locations.

Let $G(s)$ be analytic except at s_0 . Then we may write $G(s)$ in its Laurent series expansion form:

$$G(s) = \frac{A_{-n}}{(s-s_0)^n} + \dots + \frac{A_{-1}}{(s-s_0)} + B_0 + B_1(s-s_0) + \dots \quad (\text{WA.37})$$

The coefficient A_{-1} is called the **residue** of $G(s)$ at s_0 , and may be evaluated as

$$A_{-1} = \text{Res}[G(s); s_0] = \frac{1}{2\pi j} \oint_C G(s) ds, \quad (\text{WA.38})$$

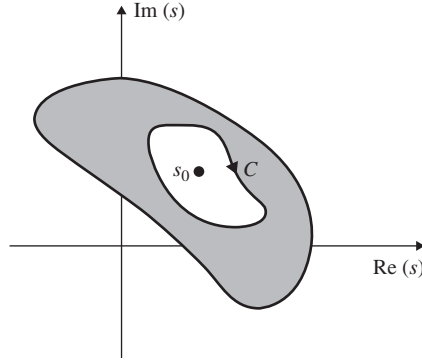
where C denotes a closed arc within an analytic region centered at s_0 that contains no other singularity, as shown in Fig. WA.6. When s_0 is not repeated with $n = 1$, we have

$$A_{-1} = \text{Res}[G(s); s_0] = (s-s_0)G(s)|_{s=s_0}. \quad (\text{WA.39})$$

This is the familiar cover-up method of computing residues.

Figure WA.6

Contour around an isolated singularity



WA.9 Residue Theorem

If the contour C contains l singularities, then Eq. (WA.39) may be generalized to yield **Cauchy's residue theorem**:

$$\frac{1}{2\pi j} \oint G(s) ds = \sum_{i=1}^l \text{Res}[G(s); s_i]. \quad (\text{WA.40})$$

WA.10 The Argument Principle

Before stating the argument principle, we need a preliminary result from which the principle follows readily.

Number of Poles and Zeros

Let $G(s)$ be an analytic function inside and on a closed contour C , except for a finite number of poles inside C . Then, for C described in the positive sense (clockwise direction),

$$\frac{1}{2\pi j} \oint \frac{G'(s)}{G(s)} ds = N - P, \quad (\text{WA.41})$$

or

$$\frac{1}{2\pi j} \oint d(\ln G) = N - P, \quad (\text{WA.42})$$

where N and P are the total number of zeros and poles of G inside C , respectively. A pole or zero of multiplicity k is counted k times.

Proof Let s_0 be a zero of G with multiplicity k . Then, in some neighborhood of that point, we may write $G(s)$ as

$$G(s) = (s - s_0)^k f(s), \quad (\text{WA.43})$$

where $f(s)$ is analytic and $f(s_0) \neq 0$. If we differentiate Eq. (WA.43), we obtain

$$G'(s) = k(s - s_0)^{k-1}f(s) + (s - s_0)^k f'(s). \quad (\text{WA.44})$$

Equation (WA.44) may be rewritten as

$$\frac{G'(s)}{G(s)} = \frac{k}{s - s_0} + \frac{f'(s)}{f(s)}. \quad (\text{WA.45})$$

Therefore, $G'(s)/G(s)$ has a pole at $s = s_0$ with residue K . This analysis may be repeated for every zero. Hence, the sum of the residues of $G'(s)/G(s)$ is the number of zeros of $G(s)$ inside C . If s_0 is a pole with multiplicity l , we may write

$$h(s) = (s - s_0)^l G(s), \quad (\text{WA.46})$$

where $h(s)$ is analytic and $h(s_0) \neq 0$. Then Eq. (WA.46) may be rewritten as

$$G(s) = \frac{h(s)}{(s - s_0)^l}. \quad (\text{WA.47})$$

Differentiating Eq. (WA.47), we obtain

$$G'(s) = \frac{h'(s)}{(s - s_0)^l} - \frac{lh(s)}{(s - s_0)^{l+1}}, \quad (\text{WA.48})$$

so

$$\frac{G'(s)}{G(s)} = \frac{-l}{s - s_0} + \frac{h'(s)}{h(s)}. \quad (\text{WA.49})$$

This analysis may be repeated for every pole. The result is that the sum of the residues of $G'(s)/G(s)$ at all the poles of $G(s)$ is $-P$.

The Argument Principle

Using Eq. (WA.38), we get

$$\frac{1}{2\pi j} \oint_C d[\ln G(s)] = N - P, \quad (\text{WA.50})$$

where $d[\ln G(s)]$ was substituted for $G'(s)/G(s)$. If we write $G(s)$ in polar form, then

$$\begin{aligned} \oint_{\Gamma} d[\ln G(s)] &= \oint_{\Gamma} d\{\ln |G(s)| + j \arg[\ln G(s)]\} \\ &= \ln |G(s)|_{s=s_1}^{s=s_2} + j \arg G(s)_{s=s_1}^{s=s_2}. \end{aligned} \quad (\text{WA.51})$$

Because Γ is a closed contour, the first term is zero, but the second term is 2π times the net encirclements of the origin:

$$\frac{1}{2\pi j} \oint_{\Gamma} d[\ln G(s)] = N - P. \quad (\text{WA.52})$$

Intuitively, the argument principle may be stated as follows: We let $G(s)$ be a rational function that is analytic except possibly at a finite number of points. We select an arbitrary contour in the s -plane so $G(s)$

is analytic at every point on the contour (the contour does not pass through any of the singularities). The corresponding mapping into the $G(s)$ -plane may encircle the origin. The number of times it does so is determined by the difference between the number of zeros and the number of poles of $G(s)$ encircled by the s -plane contour. The direction of this encirclement is determined by which is greater, N (clockwise) or P (counter-clockwise). For example, if the contour encircles a single zero, the mapping will encircle the origin once in the clockwise direction. Similarly, if the contour encloses only a single pole, the mapping will encircle the origin, this time in the counter-clockwise direction. If the contour encircles no singularities, or if the contour encloses an equal number of poles and zeros, there will be no encirclement of the origin. A contour evaluation of $G(s)$ will encircle the origin if there is a nonzero net difference between the encircled singularities. The mapping is **conformal** as well, which means that the magnitude and sense of the angles between smooth arcs is preserved. Chapter 6 provides a more detailed intuitive treatment of the argument principle and its application to feedback control in the form of the Nyquist stability theorem.

WA.11 Bilinear Transformation

A bilinear transformation is of the form

$$w = \frac{as + b}{cs + d}, \quad (\text{WA.53})$$

where a, b, c, d are complex constants, and it is assumed $ad - bc \neq 0$. The bilinear transformation always transforms circles in the w -plane into circles in the s -plane. This can be shown in several ways. If we solve for s , we obtain

$$s = \frac{-dw + b}{cw - a}. \quad (\text{WA.54})$$

The equation for a circle in the w -plane is of the form

$$\frac{|w - \sigma|}{|w - \rho|} = R. \quad (\text{WA.55})$$

If we substitute for w in terms of s in Eq. (WA.53), we get

$$\frac{|s - \sigma'|}{|s - \rho'|} = R', \quad (\text{WA.56})$$

where

$$\sigma' = \frac{\sigma d - b}{a - \sigma c}, \quad \rho' = \frac{\rho d - b}{a - \rho c}, \quad R' = \left| \frac{a - \rho c}{a - \sigma c} \right| R, \quad (\text{WA.57})$$

which is the equation for a circle in the s -plane. For alternative proofs, the reader is referred to Brown and Churchill (1996) and Marsden and Hoffman (1998).