

Appendix W3.8

Amplitude and Time Scaling

The magnitude of the values of the variables in a problem is often very different, sometimes so much so that numerical difficulties arise. This was a serious problem years ago when equations were solved using analog computers, and it was routine to *scale* the variables so all had similar magnitudes. Today's widespread use of digital computers for solving differential equations has largely eliminated the need to scale a problem unless the number of variables is very large, because computers are now capable of accurately handling numbers with wide variations in magnitude. Nevertheless, it is wise to understand the principle of scaling for the few cases in which extreme variations in magnitude exist, and scaling is necessary or the computer word size is limited.

W3.8.1 Amplitude Scaling

There are two types of scaling that are sometimes carried out: amplitude scaling and time scaling, as we have already seen in Section 3.1.4. **Amplitude scaling** is usually performed unwittingly by simply picking units that make sense for the problem at hand. For the ball levitator, expressing the motion in millimeters and the current in milliamps would keep the numbers within a range that is easy to work with. Equations of motion are sometimes developed in the standard SI units such as meters, kilograms, and amperes, but when computing the motion of a rocket going into orbit, using kilometers makes more sense. The equations of motion are usually solved using computer-aided design software, which is often capable of working in any units. For higher-order systems, it becomes important to scale the problem so that system variables have similar numerical variations. A method for accomplishing the best scaling for a complex system is first to estimate the maximum values for each system variable, then to scale the system so that each variable varies between -1 and 1 .

In general, we can perform amplitude scaling by defining the scaled variables for each state element. If

$$x' = S_x x, \quad (\text{W3.6})$$

then

$$\dot{x}' = S_x \dot{x} \quad \text{and} \quad \ddot{x}' = S_x \ddot{x}. \quad (\text{W3.7})$$

We then pick S_x to result in the appropriate scale change, substitute Eqs. (W3.6) and (W3.7) into the equations of motion, and recompute the coefficients.

EXAMPLE W3.7

Scaling for the Ball Levitator

The linearized equation of motion for the ball levitator (see Example 9.2, Chapter 9) is

$$\delta\ddot{x} = 1667\delta x + 47.6\delta i, \quad (\text{W3.8})$$

where δx is in units of meters and δi is in units of amperes. Scale the variables for the ball levitator to result in units of millimeters and milliamps instead of meters and amps.

Solution. Referring to Eq. (W3.6), we define

$$\delta x' = S_x \delta x \quad \text{and} \quad \delta i' = S_i \delta i,$$

such that both S_x and S_i have a value of 1000 in order to convert δx and δi in meters and amps to $\delta x'$ and $\delta i'$ in millimeters and milliamps. Substituting these relations into Eq. (W3.8) and taking note of Eq. (W3.7) yields

$$\delta\ddot{x}' = 1667\delta x' + 47.6 \frac{S_x}{S_i} \delta i'.$$

In this case, $S_x = S_i$, so Eq. (W3.8) remains unchanged. Had we scaled the two quantities by different amounts, there would have been a change in the last coefficient in the equation.

W3.8.2 Time Scaling

The unit of time when using SI units or English units is seconds. Computer-aided design software is *usually* able to compute results accurately no matter how fast or slow the particular problem at hand. However, if a dynamic system responds in a few microseconds, or if there are characteristic frequencies in the system on the order of several megahertz, the problem may become ill-conditioned, so that the numerical routines produce errors. This can be particularly troublesome for high-order systems. The same holds true for an extremely slow system. It is therefore useful to know how to change the units of time should you encounter an ill-conditioned problem.

We define the new scaled time to be

$$\tau = \omega_o t, \quad (\text{W3.9})$$

such that, if t is measured in seconds and $\omega_o = 1000$, then τ will be measured in milliseconds. The effect of the **time scaling** is to change the differentiation so

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d(\tau/\omega_o)} = \omega_o \frac{dx}{d\tau}, \quad (\text{W3.10})$$

and

$$\ddot{x} = \frac{d^2x}{dt^2} = \omega_o^2 \frac{d^2x}{d\tau^2}. \quad (\text{W3.11})$$

EXAMPLE W3.8

Time Scaling an Oscillator

The equation for an oscillator was derived in Example 2.6. For a case with a very fast natural frequency $\omega_n = 15,000$ rad/sec (about 2 kHz), Eq. (2.23) can be rewritten as

$$\ddot{\theta} + 15,000^2 \cdot \theta = 10^6 \cdot T_c.$$

Determine the time-scaled equation so that the unit of time is milliseconds.

Solution. The value of ω_o in Eq. (W3.9) is 1000. Equation (W3.11) shows that

$$\frac{d^2\theta}{d\tau^2} = 10^{-6} \cdot \ddot{\theta},$$

and the time-scaled equation becomes

$$\frac{d^2\theta}{d\tau^2} + 15^2 \cdot \theta = T_c.$$

In practice, we would then solve the equation

$$\ddot{\theta} + 15^2 \cdot \theta = T_c \quad (\text{W3.12})$$

and label the plots in milliseconds instead of seconds.

W3.8.3 Time and Amplitude Scaling in State-Space

We have already discussed time and amplitude scaling in Chapter 3. We now extend the ideas to the state-variable form. Time scaling with $\tau = \omega_o t$ replaces Eq. (7.3) with

$$\frac{d\mathbf{x}}{d\tau} = \frac{1}{\omega_o} \mathbf{A}\mathbf{x} + \frac{1}{\omega_o} \mathbf{B}u = \hat{\mathbf{A}}\mathbf{x} + \hat{\mathbf{B}}u. \quad (\text{W3.13})$$

Amplitude scaling of the state corresponds to replacing \mathbf{x} with $\mathbf{z} = \mathbf{D}_x^{-1}\mathbf{x}$, where \mathbf{D}_x is a diagonal matrix of scale factors. Input scaling corresponds to replacing u with $v = \mathbf{D}_u^{-1}u$. With these substitutions,

$$\mathbf{D}_x \dot{\mathbf{z}} = \frac{1}{\omega_o} \mathbf{A} \mathbf{D}_x \mathbf{z} + \frac{1}{\omega_o} \mathbf{B} \mathbf{D}_u v. \quad (\text{W3.14})$$

Then

$$\dot{\mathbf{z}} = \frac{1}{\omega_o} \mathbf{D}_x^{-1} \mathbf{A} \mathbf{D}_x \mathbf{z} + \frac{1}{\omega_o} \mathbf{D}_x^{-1} \mathbf{B} \mathbf{D}_u v = \hat{\mathbf{A}}\mathbf{z} + \hat{\mathbf{B}}v. \quad (\text{W3.15})$$

Equation (W3.15) compactly expresses the time- and amplitude-scaling operations. Regrettably, it does not relieve the engineer of the responsibility of actually thinking of good scale factors so scaled equations are in good shape.

EXAMPLE W3.9 *Time Scaling an Oscillator*

The equation for an oscillator was derived in Example 2.6. For a case with a very fast natural frequency $\omega_n = 15,000$ rad/sec (about 2 kHz), Eq. (2.23) can be rewritten as

$$\ddot{\theta} + 15,000^2 \cdot \theta = 10^6 \cdot T_c.$$

Determine the time-scaled equation so the unit of time is milliseconds.

Solution. In state-variable form with a state vector $\mathbf{x} = [\theta \ \dot{\theta}]^T$, the unscaled matrices are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -15,000^2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 10^6 \end{bmatrix}.$$

Applying Eq. (W3.13) results in

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & \frac{1}{1000} \\ -\frac{15,000^2}{1000} & 0 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{B}} = \begin{bmatrix} 0 \\ 10^3 \end{bmatrix},$$

which yields state-variable equations that are scaled.
