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Linear Multivariable Servomechanisms Revisited: System Type and Accuracy Trade-offs*

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Key Words—Servomechanisms; overshoot; linear multivariable systems; system type; tracking characteristics.

Abstract—It is shown that every scalar closed-loop output of a linear multivariable type m servomechanism, where m is larger than one, will always track its corresponding reference trajectory with an instantaneous tracking error whose integral over time identically vanishes. This is true for all polynomial reference signals of up to order $m-2$. The simplest implication of this result is that the step responses of a multivariable servomechanism, which is capable of asymptotically following ramp signals, must all display overshoot. This is analogous to the familiar classical control result. New insight into tracking of dependent signals is also provided using the notion of "generalized" system type. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

In scalar servomechanism theory, e.g. see Franklin *et al.* (1994), system type is a well-established concept for assessing the asymptotic (polynomial) tracking error properties of a closed-loop system from an examination of its open-loop transfer function. It is also generally accepted that two criteria of goodness used for the problem of designing a servomechanism are the degree of asymptotic accuracy and the quality of the transient response. Nevertheless, only recently (León de la Barra and Fernández, 1994), the transient tracking error properties of type m scalar servomechanisms have been the subject of rigorous attention. This paper generalizes the results in León de la Barra and Fernández (1994) to the multivariable setting, simultaneously providing new insight into a "generalized" system type notion available in the literature (Emami-Naeini, 1981), as well as into the input–output structure of linear multivariable servomechanisms.

It is known, e.g. see Sebakhy (1984), Wolfe and Meditch (1977), and Zhang (1986), that a bounded input bounded output (BIBO) stable type $[m_1\ m_2\ \cdots\ m_n]$ linear multivariable servomechanism tracks with zero asymptotic error all polynomial vector reference signals of up to order $[m_1-1\ m_2-1\ \cdots\ m_n-1]$, $m_i \geq 1$, $i=1,\ldots,n$.

This paper shows that if $m_j \geq 2$, the jth closed-loop output of a BIBO stable type $[m_1\ m_2\ \cdots\ m_n]$ linear multivariable servomechanism will track every polynomial reference input of up to order m_j-2 with an instantaneous tracking error whose integral over the interval $[0,+\infty)$ identically vanishes. This determines that the tracking error cannot have a single sign for all values of time, and that there will necessarily be overshoot in the response of the jth output.

The paper also introduces a number of new results which provide a deeper understanding of the mechanisms involved in asymptotically tracking (polynomial) vector input signals.

2. *Preliminaries*

We consider BIBO stable type $[m_1\ m_2\ \cdots\ m_n]$ linear multivariable servomechanisms, as the one shown in Fig. 1, whose closed-loop reference-to-output transfer function matrix is assumed to be rational, proper and given by

$$\mathbf{G}(s) \doteq [\mathbf{g}_1(s)\quad \mathbf{g}_2(s)\quad \cdots\quad \mathbf{g}_n(s)] = [g_{i,j}(s)]\,,$$

where $\mathbf{g}_i(s)$ denotes the ith column of $\mathbf{G}(s)$, and $g_{i,j}(s)$ corresponds to its i,j entry, respectively. Note that $\mathbf{G}(s)$ is explicitly given by $\mathbf{G}(s)=\{\mathbf{I}_n+\mathbf{L}(s)\}^{-1}\mathbf{L}(s)$, where $\mathbf{L}(s)$ is the open-loop transfer function matrix, and $\mathbf{I}_n$ is the identity matrix of size n. It is also assumed that $\mathbf{L}(s)$ is free of hidden unstable modes.

Let us also introduce a family of elementary n-dimensional vector reference trajectories given by

$$\mathbf{r}_{i,j}(t) \doteq \Big(0,\ldots,0,\overbrace{t^j}^{i\text{th column}},0,\ldots,0\Big)^{\mathrm{T}},\quad t\geq 0\,,$$

where $1\leq i\leq n$, $j\geq 0$, and the superscript T denotes transposition. Likewise, $\mathbf{y}_{i,j}(t)\in\mathbb{R}^n$, where $\mathbb{R}$ is the set of real numbers, denotes the output of $\mathbf{G}(s)$ when its input is $\mathbf{r}_{i,j}(t)$, and $\mathbf{e}_{i,j}(t)\doteq\mathbf{r}_{i,j}(t)-\mathbf{y}_{i,j}(t)$ denotes the instantaneous tracking error signal, respectively. The following definition is taken from Sebakhy (1984), Wolfe and Meditch (1977) and Zhang (1986).

Definition 1. A unity feedback multivariable linear system is called type $[m_1\ m_2\ \cdots\ m_n]$ if its open-loop transfer function matrix $\mathbf{L}(s)$ can be written in the form

$$\mathbf{L}(s)=\mathbf{H}(s)\cdot\tilde{\mathbf{L}}(s)\,, \tag{1}$$

where

$$\mathbf{H}(s)\doteq\mathrm{diag}\left(\frac{1}{s^{m_1}}\ \frac{1}{s^{m_2}}\ \cdots\ \frac{1}{s^{m_n}}\right),\quad m_i\in\mathbb{Z},\ i=1,\ldots,n\,, \tag{2}$$

and

$$\lim_{s\to 0}\{\Delta_{\tilde{L}}(s)\cdot\det[\tilde{\mathbf{L}}(s)]\}\neq 0\,, \tag{3}$$

where $\mathbb{Z}$ is the set of integer numbers, and $\Delta_{\tilde{L}}(s)$ is the characteristic polynomial of $\tilde{\mathbf{L}}(s)$. Here, $\mathbf{L}(s)$ is referred to as a type $[m_1\ m_2\ \cdots\ m_n]$ transfer function matrix.

Note that the n-tuple $[m_1\ m_2\ \cdots\ m_n]$ introduced in Definition 1 is uniquely determined by conditions (1)–(3) (Sebakhy, 1984). Intuitively, this means that a chain of integrators of degree $[m_1\ m_2\ \cdots\ m_n]$ can be isolated at the output of $\mathbf{L}(s)$.

Let us also introduce the following sets: $J_0^-\doteq\{i|m_i\leq 0\}$, $J_0^+\doteq\{i|m_i\geq 0\}$, $J_1\doteq\{i|m_i\geq 1\}$, and $J_2\doteq\{i|m_i\geq 2\}$, i.e. J_0^-, J_0^+,

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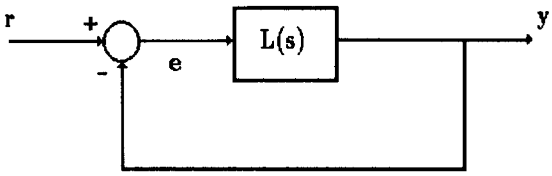

Fig. 1. A linear multivariable servomechanism.

J_1, and J_2 are associated to those closed-loop outputs whose type is non-positive, non-negative, at least one, and at least two, respectively.

Theorem 1. Let $\mathbf{L}(s)$ be a type $[m_1\ m_2\ \cdots\ m_n]$ transfer function matrix. Then the BIBO stable unity feedback system having $\mathbf{L}(s)$ as its open-loop transfer function matrix will track without asymptotic error every polynomial vector reference input of the form

$$\mathbf{r}(t) \doteq \sum_{i \in J_1} \sum_{j=0}^{m_i - 1} r_{i,j} \cdot \mathbf{r}_{i,j}(t)$$

$$= \sum_{i \in J_1} \begin{array}{r} \\ \\ \\ \text{ith row} \\ \\ \\ \\ \end{array} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r_{i,0} + r_{i,1} \cdot t + \cdots + r_{i,m_i - 1} \cdot t^{m_i - 1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4}$$

where the coefficients $r_{i,j}$ can vary independently for $i \in J_1$, and $0 \le j \le m_i - 1$.

Proof. See Sebakhy (1984) and Wolfe and Meditch (1977). □

Note that in equation (4) $\mathbf{r}(t) \doteq [r_1(t)\ r_2(t\ \cdots\ r_n(t)]^{\mathrm{T}}$, $t \ge 0$, is an $n \times 1$ polynomial vector whose kth row is identically zero for all time and for all $k \in J_0^-$, i.e. for all those closed-loop outputs which have non-positive type.

3. *Main results*

3.1. *A connection between* $\mathbf{g}_i(s)$ *and* $\mathbf{e}_{i,j}(t)$. The following lemma relates the behaviour of the columns of $\mathbf{G}(s)$ at $s = 0$ to the asymptotic behaviour of the error $\mathbf{e}_{i,j}(t)$.

Lemma 1. Let $\mathbf{G}(s)$ be the closed-loop reference-to-output transfer function matrix of a BIBO stable type $[m_1\ m_2\ \cdots\ m_n]$ servomechanism. It then follows that $\mathbf{e}_{i,j}(t) = \mathbf{r}_{i,j}(t) - \mathbf{y}_{i,j}(t)$ satisfies

$$\mathbf{e}_{i,0}(+\infty) = \boldsymbol{\delta}_i - \mathbf{g}_i(0) \ne \mathbf{0}, \quad i \in J_0^-, \tag{5}$$

$$\mathbf{e}_{i,0}(+\infty) = \mathbf{0}, \quad \mathbf{e}_{i,m_i}(+\infty) = -\mathbf{g}_i^{(m_i)}(0) \ne \mathbf{0}, \quad i \in J_1, \tag{6}$$

$$\mathbf{e}_{i,j}(+\infty) = -\mathbf{g}_i^{(j)}(0) = \mathbf{0}, \quad i \in J_2,\ 1 \le j \le m_i - 1, \tag{7}$$

where $\boldsymbol{\delta}_i$ is the ith column of $\mathbf{I}_n$, $\mathbf{0}$ is the null vector, and the superscript (j) denotes derivative of order j.

Proof. It can be easily seen that the Laplace transform of $\mathbf{e}_{i,j}(t)$, denoted by $\mathscr{L}[\mathbf{e}_{i,j}(t)] = \mathbf{E}_{i,j}(s)$, is given by

$$\mathbf{E}_{i,j}(s) = \{\mathbf{I}_n - \mathbf{G}(s)\} \cdot \left(0, \ldots, 0, \overbrace{\frac{j!}{s^{j+1}}}^{\text{ith column}}, 0, \ldots, 0 \right)^{\mathrm{T}}$$

$$= \frac{j!\{\boldsymbol{\delta}_i - \mathbf{g}_i(s)\}}{s^{j+1}}, \tag{8}$$

and using the Final Value Theorem leads to

$$\mathbf{e}_{i,j}(+\infty) = \lim_{s \to 0} \{s \cdot \mathbf{E}_{i,j}(s)\} = j! \lim_{s \to 0} \left\{ \frac{\boldsymbol{\delta}_i - \mathbf{g}_i(s)}{s^j} \right\}.$$

If $j = 0$, it follows directly from the asymptotic tracking error properties described in Theorem 1 that

$$\mathbf{e}_{i,0}(+\infty) = \lim_{s \to 0} \{\boldsymbol{\delta}_i - \mathbf{g}_i(s)\} = \begin{cases} \boldsymbol{\delta}_i - \mathbf{g}_i(0) \ne \mathbf{0}, & i \in J_0^-, \\ \mathbf{0}, & i \in J_1, \end{cases} \tag{9}$$

i.e. $\mathbf{g}_i(0) = \boldsymbol{\delta}_i$ for $i \in J_1$.

Likewise, if $i \in J_2$ it follows from Theorem 1 and equation (9), by applying L'Hôpital's rule, that

$$\mathbf{e}_{i,j}(+\infty) = j! \lim_{s \to 0} \left\{ \frac{\boldsymbol{\delta}_i - \mathbf{g}_i(s)}{s^j} \right\} = -\mathbf{g}_i^{(j)}(0) = \mathbf{0}, \quad 1 \le j \le m_i - 1.$$

Note also that Theorem 1 establishes that for $i \in J_1$, $\mathbf{e}_{i,m_i}(+\infty)$ is both finite and non-zero, i.e.

$$\mathbf{e}_{i,m_i}(+\infty) = m_i! \lim_{s \to 0} \left\{ \frac{\boldsymbol{\delta}_i - \mathbf{g}_i(s)}{s^{m_i}} \right\} = -\mathbf{g}_i^{(m_i)}(0) \ne \mathbf{0}.$$

It can also be seen that $\mathbf{e}_{i,j}(+\infty) \to \infty$ for $j > m_i$. □

The following corollary relates Lemma 1 to the behaviour of the columns of the closed-loop reference-to-error transfer function matrix at $s = 0$.

Corollary 1. $G(s)$ is the closed-loop reference-to-output transfer function matrix of a BIBO stable type $[m_1\ m_2\ \cdots\ m_n]$ servomechanism iff $\boldsymbol{\delta}_i - \mathbf{g}_i(s)$, $i \in J_0^+$, has exactly m_i zeros at $s = 0$.

Proof. Direct from (the proof of) Lemma 1. □

Remark 1. Note that the m_i zeros at $s = 0$ in the ith column of $\mathbf{I}_n - \mathbf{G}(s)$ are necessary and sufficient to guarantee that the contribution of $r_i(t)$ to the tracking error vector $\mathbf{e}(t) \doteq \mathbf{r}(t) - \mathbf{y}(t)$ asymptotically goes to zero if $\mathscr{L}[r_i(t)] = \sum_{j=0}^{m_i - 1} j! \cdot r_{i,j}\ s^{m_i - j - 1}/s^{m_i}$, $i \in J_1$, cf. equation (4) for the corresponding time domain expression.

3.2. *Tracking of higher order dependent polynomial trajectories.* The following proposition highlights how the overall asymptotic tracking error $\mathbf{e}(t)$ decomposes into errors due to each one of the components of a general reference input signal $\mathbf{r}(t)$.

Proposition 1. Let $\mathbf{G}(s)$ be the closed-loop reference-to-output transfer function matrix of a BIBO stable type $[m_1\ m_2\ \cdots\ m_n]$ servomechanism, and let $\mathbf{y}(t)$ denote its response to $\mathbf{r}(t) \doteq \sum_{i \in J_0^+} \sum_{j=0}^{m_i} r_{i,j} \cdot \mathbf{r}_{i,j}(t)$. It then follows that the tracking error $\mathbf{e}(t)$ has the asymptotic value

$$\mathbf{e}(+\infty) = \sum_{i \in J_0^+} r_{i,m_i} \cdot \mathbf{e}_{i,m_i}(+\infty)$$

$$= \sum_{\{i \mid m_i = 0\}} r_{i,0} \cdot \{\boldsymbol{\delta}_i - \mathbf{g}_i(0)\} - \sum_{i \in J_1} r_{i,m_i} \cdot \mathbf{g}_i^{(m_i)}(0). \tag{10}$$

Proof. Since $\mathbf{e}(t)$ is explicitly given by

$$\mathbf{e}(t) = \sum_{i \in J_0^+} \sum_{j=0}^{m_i} r_{i,j} \cdot \{\mathbf{r}_{i,j}(t) - \mathbf{y}_{i,j}(t)\} = \sum_{i \in J_0^+} \sum_{j=0}^{m_i} r_{i,j} \cdot \mathbf{e}_{i,j}(t),$$

the result follows directly from Lemma 1. □

Remark 2. Note that in equation (10) we have that $\mathbf{e}_{i,m_i}(+\infty) \ne \mathbf{0}$ for all $i \in J_0^+$. However, if the vectors $\mathbf{e}_{i,m_i}(+\infty)$, $i \in J_0^+$, were linearly dependent, $\mathbf{e}(+\infty)$ could become zero for a non-trivial selection of the coefficients r_{i,m_i}, $i \in J_0^+$. Obviously, this does not have counterpart in the scalar setting. It is in this sense that a type $[m_1\ m_2\ \cdots\ m_n]$ servomechanism may be able to follow, with no asymptotic error, higher order dependent polynomial trajectories (Emami-Naeini, 1981). Thus, the standard system type definition, cf. Definition 1, does not fully describe the tracking capability of multivariable systems with respect to polynomial time functions. The following example illustrates this issue.

3.2.1. *Example* 1. Let us assume that the closed-loop reference-to-output transfer function matrix of a BIBO stable type [1 1] servomechanism is given by

$$\mathbf{G}(s) \doteq \begin{pmatrix} \dfrac{2}{s+2} & \dfrac{s}{(s+1)^2} \\ \dfrac{s}{(s+2)^2} & \dfrac{2}{s+2} \end{pmatrix} = [\mathbf{g}_1(s) \quad \mathbf{g}_2(s)]. \tag{11}$$

From Proposition 1 it follows that $\mathbf{G}(s)$ will exhibit zero asymptotic error for all first-order dependent trajectories

$$\mathbf{r}(t) \doteq \begin{pmatrix} r_{1,0} + r_{1,1} \cdot t \\ r_{2,0} + r_{2,1} \cdot t \end{pmatrix}$$

satisfying

$$r_{1,1} \cdot \mathbf{g}_1^{(1)}(0) + r_{2,1} \cdot \mathbf{g}_2^{(1)}(0) = r_{1,1} \cdot \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{pmatrix} + r_{2,1} \cdot \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} = \mathbf{0}.$$

In other words, the above (standard) type [1 1] servomechanism will display zero asymptotic error for every first-order reference signal having the form

$$\mathbf{r}(t) \doteq \begin{pmatrix} r_{1,0} + r_{1,1} \cdot t \\ r_{2,0} + \dfrac{r_{1,1}}{2} \cdot t \end{pmatrix}, \quad r_{i,j} \in \mathbb{R}.$$

Thus, the servomechanism whose closed-loop reference-to-output transfer function matrix is given by equation (11) is said to be a "generalized" type [2 2] servomechanism because of its capacity to follow (a restricted class of) first-order trajectories without asymptotic error (Emami-Naeini, 1981).

3.3. *A trade-off between asymptotic and transient accuracies.* The following result connects $\mathbf{e}_{i,j+1}(+\infty)$ to the tracking error $\mathbf{e}_{i,j}(t)$.

Theorem 2. Let $\mathbf{G}(s)$ be the closed-loop reference-to-output transfer function matrix of a BIBO stable type $[m_1 \ m_2 \ \cdots \ m_n]$ servomechanism. It then follows that

$$\mathbf{e}_{i,j+1}(+\infty) = (j+1) \int_0^{+\infty} \mathbf{e}_{i,j}(t)\,\mathrm{d}t, \tag{12}$$

for $i \in J_1$, and $0 \le j \le m_i - 1$.

Proof. It is clear from equation (8) that

$$\mathscr{L}[\mathbf{e}_{i,j+1}(t)] = \mathbf{E}_{i,j+1}(s) = \frac{(j+1)!\{\boldsymbol{\delta}_i - \mathbf{g}_i(s)\}}{s^{j+2}},$$

and using the Final Value Theorem leads to

$$\begin{aligned} \mathbf{e}_{i,j+1}(+\infty) &= \lim_{s\to 0}\{s \cdot \mathbf{E}_{i,j+1}(s)\} = (j+1)\lim_{s\to 0}\{\mathbf{E}_{i,j}(s)\} \\ &= (j+1)\lim_{s\to 0}\{\mathscr{L}[\mathbf{e}_{i,j}(t)]\} \\ &= (j+1)\lim_{s\to 0}\left\{\int_0^{+\infty} \mathbf{e}_{i,j}(t)\,\mathrm{e}^{-st}\,\mathrm{d}t\right\} \end{aligned} \tag{13}$$

and the result follows by making $s = 0$ in the above Laplace Integral.† □

Note that equation (12) constitutes the multivariable generalization of the familiar classical control statement "... the error in following a ramp is the integral of the error in tracking a step...". Theorem 3 follows readily from Lemma 1 and Theorem 2.

Theorem 3. Let $\mathbf{G}(s)$ be the closed-loop reference-to-output transfer function matrix of a BIBO stable type $[m_1 \ m_2 \ \cdots \ m_n]$ servomechanism. It then follows that

$$\int_0^{+\infty} \mathbf{e}_{i,j}(t)\,\mathrm{d}t = \int_0^{+\infty} \{\mathbf{r}_{i,j}(t) - \mathbf{y}_{i,j}(t)\}\,\mathrm{d}t = \mathbf{0}, \tag{14}$$

for $i \in J_2$, and $0 \le j \le m_i - 2$.

Theorem 3 states that $\mathbf{y}_{i,j}(t)$ determines equal magnitude areas below and above the corresponding reference input $\mathbf{r}_{i,j}(t)$. In other words, $\mathbf{y}_{i,j}(t)$ will always overshoot $\mathbf{r}_{i,j}(t)$. A straightforward generalization of Theorem 3 is stated below.

Corollary 2. Let $\mathbf{G}(s)$ be the closed-loop reference-to-output transfer function matrix of a BIBO stable type $[m_1 \ m_2 \ \cdots \ m_n]$ servomechanism with polynomial vector reference input $\mathbf{r}(t) \doteq \sum_{i\in J_2} \sum_{j=0}^{m_i-2} r_{i,j} \cdot \mathbf{r}_{i,j}(t)$ and associated output $\mathbf{y}(t)$. It follows that

$$\int_0^{+\infty} \mathbf{e}(t)\,\mathrm{d}t = \int_0^{+\infty} \{\mathbf{r}(t) - \mathbf{y}(t)\}\,\mathrm{d}t = \mathbf{0}. \tag{15}$$

Proof. This is a simple linear combination of the integrals in equation (14). □

Remark 3. Corollary 2 determines that $\mathbf{y}(t)$ defines equal magnitude areas below and above the corresponding reference input $\mathbf{r}(t)$. In other words, $\mathbf{y}(t)$ will always overshoot $\mathbf{r}(t)$. Note that overshoot during set point changes is undesirable in many practical problems, see e.g. Ebert *et al.* (1995) for a rapid thermal processing application where overshoot must be avoided. In process control, for example, the optimum set point may be close to an economic or a safety constraint. As a result, overshoot of a set point could lead to a violation of a constraint possibly endangering process operation, see Jayasuriya and Song (1996) for further motivation.

Note also that in contrast to Theorem 3, Corollary 2 admits simultaneous set-point changes for all the closed-loop outputs having type larger than one. In addition, the coefficients $r_{i,j}$ can take arbitrary values. The following example illustrates the results presented in this subsection.

3.3.1. *Example* 2. This example is borrowed from Porter and Bradshaw (1974), where a type [2 2] linear multivariable servomechanism having a closed-loop reference-to-output transfer function matrix given by

$$\mathbf{G}(s) \doteq \frac{1}{(s+1)^6} \begin{pmatrix} \dfrac{31}{36}s^4 + \dfrac{121}{6}s^3 + \dfrac{527}{36}s^2 + 6s + 1 & -\dfrac{5}{36}s^2\left(s + \dfrac{1057}{5}\right) \\ \dfrac{77}{36}s^2\left(s^2 - \dfrac{690}{77}s + \dfrac{799}{77}\right) & -9s^4 - \dfrac{1945}{36}s^3 + \dfrac{307}{36}s^2 + 6s + 1 \end{pmatrix}$$

was introduced. It is straightforward to verify that $\boldsymbol{\delta}_i - \mathbf{g}_i(s)$ has exactly m_i zeros at $s = 0$, $i = 1, 2$, cf. Corollary 1.

Figures 2 and 3 display the closed-loop response $\mathbf{y}(t) \doteq [y_1(t) \ y_2(t)]^{\mathrm{T}}$ to the polynomial vector reference input

$$\mathbf{r}(t) \doteq \mathbf{r}_{1,0}(t) + \mathbf{r}_{2,0}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \ge 0. \tag{16}$$

†Note that $s = 0$ is in the region of convergence of integral (10) since we are dealing with BIBO stable servomechanisms.

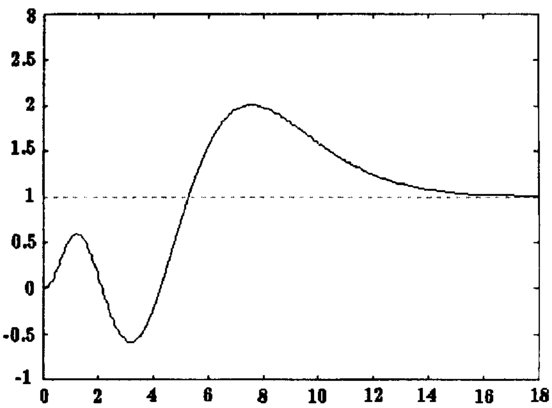

Fig. 2. First closed-loop output and reference signal.

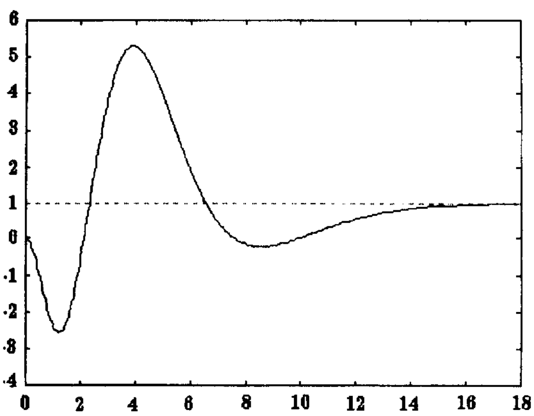

Fig. 3. Second closed-loop output and reference signal.

It is obvious from these figures that each output defines equal magnitude areas below and above its corresponding reference trajectory, see equation (15) in Corollary 2. Note also that the occurrence of overshoot is not due to complex conjugate poles in $\mathbf{G}(s)$, is it neither related to some *ad hoc* structure in $\mathbf{r}(t)$ nor in $\mathbf{G}(s)$ itself, but it is due to the inherent property predicted by Corollary 2.

A limitation of the results in this subsection is due to the lack of an estimate for the peak deviations in each of the closed-loop outputs below and above the corresponding reference trajectories. Note that in Corollary 2 the actual transient behaviour of the lth closed-loop output, $l \in J_2$, will depend upon $\{g_{l,j}(s), j \in J_2\}$, together with $\{\mathscr{L}[r_j(t)], j \in J_2\}$. This makes it hard to predict the peak deviations of the lth closed-loop output, $l \in J_2$. Nevertheless, it is possible to conjecture that the existence of complex conjugate poles in $\mathbf{G}(s)$ will tend to increase these peak deviations when compared to servomechanisms having only real closed-loop poles.

From this perspective it is obvious that the overshooting phenomenom which has been exposed in this subsection is indeed a multivariable feature, and also an essential manifestation of a trade-off between asymptotic and transient accuracies in multivariable servomechanisms.

3.4. *Characterizing the entries of* $\mathbf{G}(s)$. The proposition below characterizes the scalar entries of the closed-loop reference-to-output transfer function matrix of a BIBO stable type $[m_1\ m_2\ \cdots\ m_n]$ servomechanism.

Proposition 2. Let $\mathbf{G}(s)$ be the closed-loop reference-to-output transfer function matrix of a BIBO stable type $[m_1\ m_2\ \cdots\ m_n]$ servomechanism. It follows that

$$g_{i,i}(s) \doteq \frac{\sum_{l=0}^{n_{i,i}} d_{i,i,l} \cdot s^l}{\sum_{l=0}^{k_{i,i}} f_{i,i,l} \cdot s^l}, \quad i \in J_1, \tag{17}$$

where $d_{i,i,l} = f_{i,i,l}$, $l = 0, \ldots, m_i - 1$, $n_{i,i} \geq m_i - 1$, and

$$g_{i,j}(s) \doteq \frac{\sum_{l=m_j}^{n_{i,j}} d_{i,j,l} \cdot s^l}{\sum_{l=0}^{k_{i,j}} f_{i,j,l} \cdot s^l}, \quad i \neq j, j \in J_1 \tag{18}$$

with $d_{i,j,m_j} \neq 0$, $f_{i,j,0} \neq 0$, $n_{i,j} \geq m_j$. In other words, $g_{i,i}(s)$ is the closed-loop reference-to-output transfer function of a BIBO stable type m_i scalar servomechanism (León de la Barra and Fernández, 1994), and $g_{i,j}(s)$ has exactly m_j zeros at the origin.

Proof. It follows from Corollary 1 after a straightforward but tedious derivation which also utilizes some of the results available in León de la Barra and Fernández (1994). □

Remark 4. If $k \in J_0^-$ we can only state that $g_{l,k}(0) \neq 0$, for some $l \neq k$, or $g_{k,k}(0) \neq 1$. Remember that $\mathbf{G}(s)$ has been assumed to be both proper and BIBO stable. This obviously imposes further constraints on the values of $n_{i,i}$, $k_{i,i}$, $n_{i,j}$, $k_{i,j}$, $f_{i,i,l}$, and $f_{i,j,l}$. Nevertheless, the only constraints directly linked to positive system type are those given in equations (17) and (18).

Proposition 2 enables us to evaluate the (standard) non-negative type of a BIBO stable linear multivariable servomechanism just by inspection of the corresponding closed-loop reference-to-output transfer function matrix.

4. *Concluding remarks*

The paper has presented simple relationships between the closed-loop reference-to-output and reference-to-error transfer function matrices of a BIBO stable linear multivariable servomechanism and the asymptotic behaviour of the tracking error (Lemma 1 and Corollary 1). Insight has been given into how a BIBO stable linear multivariable servomechanism may follow without asymptotic error dependent polynomial trajectories of higher order than those defined by its (standard) type (Proposition 1). Finally, the reader is now fully aware of the fact that BIBO stable linear multivariable servomechanisms which are required to follow arbitrary polynomial trajectories of order $[m_1 - 1\ m_2 - 1\ \cdots\ m_n - 1]$ without asymptotic error will always have to exhibit overshooting lower order responses (Corollary 2). Essentially identical results are valid for discrete time multivariable servomechanisms.

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