Near-Optimal Sensor Placement for Health Monitoring of Civil Structures

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ABSTRACT

In this paper we focus on the optimal placement of sensors for state estimation-based continuous health monitoring of structures using three approaches. The first aims to minimize the static estimation error of the structure deflections, using the linear stiffness matrix derived from a finite element model. The second approach aims to maximize the observability of the derived linear state space model. The third approach aims to minimize the dynamic estimation error of the deflections using a Linear Quadratic Estimator. Both nonlinear mixed-integer and relaxed convex optimization formulations are presented. A simple search-based optimization implementation for each of the three approaches is demonstrated on a model of the long-span New Carquinez Bridge in California.

Keywords: Optimal sensor placement, Structural health monitoring, Model-based estimation, Convex optimization.

1. INTRODUCTION

The work presented in this paper is part of the NIST-TIP 2008 project: *Cyber-Enabled Wireless Monitoring Systems for the Protection of Deteriorating National Infrastructure Systems*. This $19-million project is funded by $9-million from NIST (National Institute of Standards and Technology), and is expected to run from 2009 – 2013. It is a Joint Venture (JV) led by the University of Michigan (UofM), Ann Arbor, MI, in partnership with with five private firms in Michigan, New York and California. SC Solutions, Inc. is one of the private JV partners.

This multi-disciplinary project aims to develop a comprehensive system for real-time monitoring and assessing the structural health and integrity of major civil infrastructure elements such as bridges on a regional basis, with innovations ranging in scale from 'smart material'-based sensors at the level of individual structural components up to structure-level data integration and interpretation to a Web-based system for information aggregation and decision support at the regional level. The JV is working closely with the Michigan Department of Transportation (MDOT) and the California Department of Transportation (Caltrans) in developing and testing this system.

An important task in implementing a structural health monitoring (SHM) system is determining the *number* and *locations* of the sensors. This has led to several approaches to Optimal Sensor Placement (OSP), see *e.g.* Ref. 1. There are two distinct aspects of health monitoring that influence OSP: detecting damage as early as possible, and determining the critical fatigue levels before damage occurs. The first approach can be posed as *model identification*: to detect changes in behavior of the structure as an indicator of damage. The latter approach can be posed as *state estimation*: to determine stresses and strains of the structure over time, or computation of the accumulated fatigue. As part of the project we plan to implement health monitoring systems for a long-span and a short-span bridge, which will be instrumented in collaboration with the local Department of Transportation. These systems will include both real-time fatigue monitoring and damage detection.

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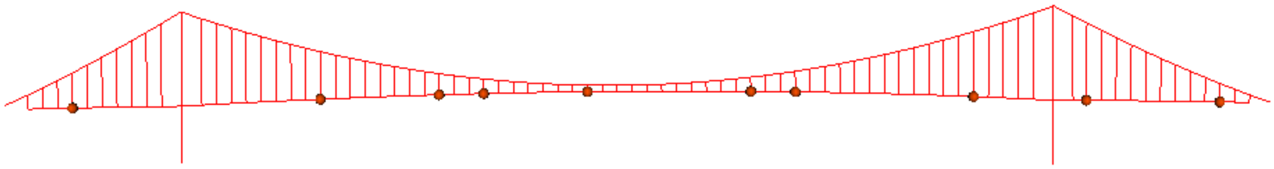

Figure 1. A schematic of the New Carquinez Bridge 2D model with potential sensor locations.

In this paper we focus on OSP for state estimation-based continuous health monitoring of structures using three approaches. The first aims to minimize the static estimation error of the structure deflections, using the linear stiffness matrix derived from a finite element (FE) model. The second approach aims to maximize the observability of the derived linear state space model. The third approach aims to minimize the dynamic estimation error of the deflections using a Linear Quadratic Estimator (LQE).

The paper is organized as follows. In Section 2 a generic bridge model is introduced. Each of the OSP approaches (static estimation, observability and dynamic estimation) is defined in Sections 3, 4 and 5 respectively. Finally, the results are presented in Section 6 and the conclusions in Section 7.

2. BRIDGE MODEL

The OSP methods described in this paper are applied to a model of the New Carquinez Bridge, which is a long span suspension bridge, managed by Caltrans (California DOT). The bridge connects the Solano and Contra Costa Counties, is located between Vallejo (on the North side) and Crockett (on the South side) in California, and has a length of 1056 m (3465 ft), with a main span of 728 m (2389 ft).

A simplified 2D finite element model of the bridge has been created that includes the anchors, towers, and deck, with a total of 270 nodes. Figure 1 shows the 2D model, with the dots indicating a potential sensor selection. It is assumed that there are 80 potential sensor locations, all on the roadbed nodes.

For the examples below the model inputs and outputs are limited to the vertical motion of the bridge deck. For the loads considered here, the bridge operates within the linear regime. Based on the finite element model locally linear static and dynamic models are constructed.

2.1 Static linear model

Assuming the dynamics are sufficiently fast, the structure can be approximated by:

$$\delta = Gf, \; G \in \mathbf{R}^{n \times p} \tag{1}$$

where G is the flexibility matrix, δ the deflection, and f the vertical forces on the roadbed. The forces f are assumed to be independent random variables:

$$f \in \mathcal{N}(0, V_f). \tag{2}$$

We assume that we can measure any of the roadbed deflections:

$$y = C_s \delta + w, \tag{3}$$

$$w \in \mathcal{N}(0, V_w) \tag{4}$$

where w is the measurement noise, C_s the sensor output matrix, y the measured output, and S the set of roadbed deflection indices.

2.2 Dynamic linear model

Assuming the dynamics are relevant, the structure can be approximated by:

$$\mathcal{M}\ddot{\delta} + \mathcal{C}\dot{\delta} + \mathcal{K}\delta = \mathcal{H}f, \;\; \mathcal{M}, \mathcal{C}, \mathcal{K} \in \mathbf{R}^{n\times n}, \;\; \mathcal{H} \in \mathbf{R}^{n\times p}, \tag{5}$$

where $\mathcal{M}$ is the mass matrix, $\mathcal{C}$ the damping matrix, $\mathcal{K}$ the stiffness matrix, δ the deflection, $\mathcal{H}$ the input matrix, and f the vertical forces on the roadbed. We assume that we can potentially measure any of the roadbed deflections. Equation (5) can be rewritten in standard state-space form:

$$\begin{aligned} \dot{x} &= Ax + Bf & (6)\\ \delta &= C_\delta x & (7)\\ y &= C_d x + w & (8) \end{aligned}$$

where:

$$\begin{aligned} x &= \begin{bmatrix} \delta \\ \dot{\delta} \end{bmatrix} & (9)\\ A &= \begin{bmatrix} 0 & I \\ -\mathcal{M}^{-1}\mathcal{K} & -\mathcal{M}^{-1}\mathcal{C} \end{bmatrix} & (10)\\ B &= \mathcal{M}^{-1}\mathcal{H} & (11)\\ C_\delta &= \begin{bmatrix} I & 0 \end{bmatrix} & (12)\\ C_d &= \begin{bmatrix} C_s & 0 \end{bmatrix} & (13)\\ f &\in \mathcal{N}(0, V_f) & (14)\\ w &\in \mathcal{N}(0, V_w) & (15) \end{aligned}$$

where w is measurement noise, x the state vector, and y the measured output.

3. STATIC ESTIMATION

Using the static linear model (see Section 2.1), the maximum a posteriori[2] (MAP) estimate of f from data y, see Equation (3), gives:

$$\hat{f} = \left(V_f^{-1} + G^T C_d^T V_w^{-1} C_d G\right)^{-1} G^T C_d^T V_w^{-1} y, \tag{16}$$

with the associated covariance:

$$\text{cov } \hat{f} = \left(V_f^{-1} + G^T C_d^T V_w^{-1} C_d G\right)^{-1}. \tag{17}$$

With $\hat{x} = G\hat{f}$, then the covariance of the state estimate is:

$$\text{cov } \hat{x} = G \; (\text{cov } \hat{f}) \; G^T = G \left(V_f^{-1} + G^T C_d^T V_w^{-1} C_d G\right)^{-1} G^T. \tag{18}$$

Introducing θ as the vector of sensor selections, where a value of 1 means that the corresponding output is selected for sensing, and 0 means that it is not selected, the static estimation-based optimal sensor selection problem can be posed as:

$$\begin{aligned} &\min_\theta \phi(\text{cov } \hat{x}(\theta)) \\ &\text{s.t.} \; \begin{cases} \theta_i \in \{0, 1\} \\ \mathbf{1}^T\theta = m \end{cases} \end{aligned} \tag{19}$$

where $\mathbf{1}$ is a vector of ones and $\phi(X)$ is a measure of the covariance size, such as $\text{Tr}(X)$, $||X||$ or $\log\det(X)$, and $C_{d,i}$ is defined as the i'th row of C_d:

$$\text{cov } \hat{x}(\theta) \;\; = \;\; = G\left(V_f^{-1} + G^T \left(\sum_{i=1}^m \theta_i C_{d,i}^T V_w^{-1} C_{d,i}\right) G\right)^{-1} G^T \tag{20}$$

which is a mixed-integer optimization problem.

3.1 Solving the mixed-integer optimization problem

The optimization problem described by Equation (19) requires an exhaustive search to find the global optimum, but a local optimum can be found using a simple search algorithm:

1. **Initialize.** Start with an empty set of selected sensors.
2. **Add most relevant sensor.** Test adding each unselected sensor to the set of selected sensors, one at a time, and keep the one giving the lowest cost.
3. **Remove redundant sensor, if any.** Test removing each of the selected sensors, one at a time, and remove the least important one if the resulting set hasn't been selected before.
4. Go back to step 2 until the maximum number of sensors is reached, or a sufficiently low cost is achieved.

This approach is applied to the static simplified model, and can also be applied to the next two approaches (see Sections 4 and 5).

3.2 Solving the relaxed convex optimization problem

Approximate solutions to Equation (19) can be obtained via *relaxation* to a convex optimization problem.[3] Convex optimization problems are readily solved using available software and computational methods.[4] Equation (19) can be converted to a relaxed sensor selection problem as follows:

$$\begin{aligned} &\min_{\theta} \phi(\text{cov } \hat{x}(\theta)) \\ &\text{s.t. } 0 \le \theta_i \le 1, \ \mathbf{1}^T\theta = m. \end{aligned} \tag{21}$$

Choosing $\phi(X) = \text{Tr}(X)$, it can be converted to the convex optimization problem:

$$\begin{aligned} &\min_{Q,\theta} \text{Tr}(Q) \\ &\text{s.t.} \begin{cases} \begin{bmatrix} Q & G \\ G^T & V_f^{-1} + \sum_i \theta_i G^T C_{d,i}^T V_w^{-1} C_{d,i} G \end{bmatrix} \ge 0 \\ 0 \le \theta_i \le 1, \\ \mathbf{1}^T\theta = m. \end{cases} \end{aligned} \tag{22}$$

The constraint $\theta_i \in \{0, 1\}$ is relaxed to allow θ_i to take any value between 0 and 1. Once the optimum is found, an approximate solution must be generated by rounding the values to either 0 and 1, see Ref. 3. The resulting optimal value of ϕ is a *lower bound* on the mixed integer problem in Equation (19). In principle, the above optimization problem can be solved using standard convex optimization tools. However, even for small finite element models the resulting problem size requires special handling, which is under development. For now, we refer to the approach in Section 3.1 as an *upper bound* on the solution that can be achieved with the convex optimization approach.

4. OBSERVABILITY

In case of dynamic estimation, the goal is to estimate the deflections from the selected sensor measurements. An important system property in this context is the observability of the system state,[5] which indicates to what extend the state can be reconstructed from the output signal.

$$\begin{aligned} &\min_{\theta} \text{Tr}(P^{-1}) \\ &\text{s.t.} \begin{cases} A^T P + PA + \sum_i \theta_i C_{d,i}^T C_{d,i} = 0, \\ P \ge 0. \end{cases} \end{aligned} \tag{23}$$

This can be solved using the search method discussed in Section 3.1.

Note that the above optimization problem is not linear or quadratic in the parameters (P and θ_i), and as a consequence cannot be readily solved using convex optimization tools. By introducing:

$$P^{-1} \leq Q \quad \Leftrightarrow \tag{24}$$

$$\begin{bmatrix} Q & I \\ I & S \end{bmatrix} \geq 0 \tag{25}$$

the problem can be rewritten as the equivalent convex optimization in P, Q and θ:

$$\begin{aligned} &\min_{\theta} \mathrm{Tr}(Q) \\ &\text{s.t.} \begin{cases} A^T P + PA + \sum_i \theta_i C_{d,i}^T C_{d,i} = 0, \\ P \geq 0, \\ \begin{bmatrix} Q & I \\ I & S \end{bmatrix} \geq 0, \\ 0 \leq \theta_i \leq 1, \\ \mathbf{1}^T \theta = m. \end{cases} \end{aligned} \tag{26}$$

5. DYNAMIC ESTIMATION

As an alternative to observability, the sensor selection problem can be directly posed as finding the set of sensors that yields the best dynamic estimator performance. Using the linear dynamic model in Equation (6) we can construct a Linear Quadratic Estimator (LQE):

$$\dot{\hat{x}} = (A - LC_d)\hat{x} + Ly \tag{27}$$

$$\hat{\delta} = C_\delta \hat{x} \tag{28}$$

where:

$$0 = AP + PA^T - P\left(\sum_i \theta_i C_{d,i}^T V_w^{-1} C_{d,i}\right) P + BV_f B^T, \tag{29}$$

$$L = P\left(\sum_i \theta_i C_d\right)^T V_w^{-1}. \tag{30}$$

P, the solution to the Riccati Equation (29), is equal to the covariance of the state estimation error, so

$$cov(\delta - \hat{\delta}) = C_\delta P C_\delta^T \tag{31}$$

As a result, the dynamic estimation-based sensor placement problem can be posed as:

$$\begin{aligned} &\min_{\theta} \phi(C_\delta P C_\delta^T) \\ &\text{s.t.} \begin{cases} AP + PA^T - P\left(\sum_i \theta_i C_{d,i}^T V_w^{-1} C_{d,i}\right) P + BV_f B^T = 0 \\ \theta_i \in \{0, 1\} \\ \mathbf{1}^T \theta = m. \end{cases} \end{aligned} \tag{32}$$

Note that Equation (29) is not linear or quadratic in the parameters (P and θ_i). By introducing:

$$S = P^{-1} \tag{33}$$

Equation (29) can be converted to an equivalent Riccati Equation:

$$0 = A^T S + SA - \sum_i \theta_i C_{d,i}^T V_w^{-1} C_{d,i} + SBV_f B^T S, \tag{34}$$

which is quadratic in the parameters. In turn, this can be rewritten as a Linear Matrix Inequality (LMI),[6] which is linear in the parameters (S and θ_i):

$$\begin{bmatrix} A^T S + SA - \sum_i \theta_i C_{d,i}^T V_w^{-1} C_{d,i} & SB \\ B^T S & -V_f^{-1} \end{bmatrix} \geq 0, \tag{35}$$

$$S \geq 0. \tag{36}$$

The sensor placement can be formulated as minimization of the estimation error covariance defined in Equation (31), leading to the following optimization problem:

$$\begin{aligned} &\min_{S,\theta} \operatorname{Tr}(C_\delta S^{-1} C_\delta^T) \\ &\text{s.t.} \begin{cases} \begin{bmatrix} -A^T S - SA + \sum_i \theta_i C_{d,i}^T V_w^{-1} C_{d,i} & SB \\ B^T S & V_f^{-1}, \end{bmatrix} \geq 0, \\ S \geq 0. \end{cases} \end{aligned} \tag{37}$$

This can be rewritten as the convex optimization problem in the variables S, Q and θ:

$$\begin{aligned} &\min_{S,Q,\theta} \operatorname{Tr}(Q) \\ &\text{s.t.} \begin{cases} \begin{bmatrix} -A^T S - SA + \sum_i \theta_i C_{d,i}^T V_w^{-1} C_{d,i} & SB \\ B^T S & V_f^{-1} \end{bmatrix} \geq 0, \\ \begin{bmatrix} Q & C_\delta \\ C_\delta^T & S \end{bmatrix} \geq 0, \\ 0 \leq \theta_i \leq 1, \\ \mathbf{1}^T \theta = m. \end{cases} \end{aligned} \tag{38}$$

6. RESULTS

The sensor selections have been determined for the simplified 2D model of the New Carquinez Bridge discussed in Section 2. For each sensor placement approach the results are shown in a plot with the cost (vertical axis) as a function of the number of selected sensors (horizontal axis), and a schematic depiction of location of the selected sensors (horizontal axis) as a function of the number of selected sensors (vertical axis). For a given number of sensors, the dots on the horizontal line show where in that case the sensors are located. The cost graph is very helpful in deciding on the number of sensors to use, as it shows how much the cost is decreased by adding another sensor. Note that the cost for each OSP approach is defined differently, and should not be compared between figures. In all plots the case of 12 selected sensors is highlighted.

Figure 2 shows the results for the static estimation-based OSP problem (see Equation (19)), Figure 3 shows the results for the observability-based OSP problem (see Equation (23)), and Figure 4 shows the results for the static estimation-based OSP problem (see Equation (32)). Each of the figures show a similar pattern: a significant decline in cost until about 20 sensors are selected, and spread out sensor locations, although not uniformly.

Note that the static estimation OSP problem in Equation 19 involves an $n \times n$ matrix inverse, which is implemented efficiently using Gaussian elimination. The observability OSP problem in Equation 23 involves solving a $2n \times 2n$ Lyaponuv equation. Finally, the dynamic estimation OSP problem in Equation 32 involves solving a $2n \times 2n$ Riccati equation. The time to compute each of the cases on an Intel-based computer (Intel Core Duo, 2.16GHz) is shown in Table 1.

As expected the static estimation-based approach is the fastest. The observability-based approach is almost 20 times slower, and the dynamic estimation-based approach another 6 times slower. Note that these results are for a small model; increasing the finite element model size will significantly increase the required computation time.

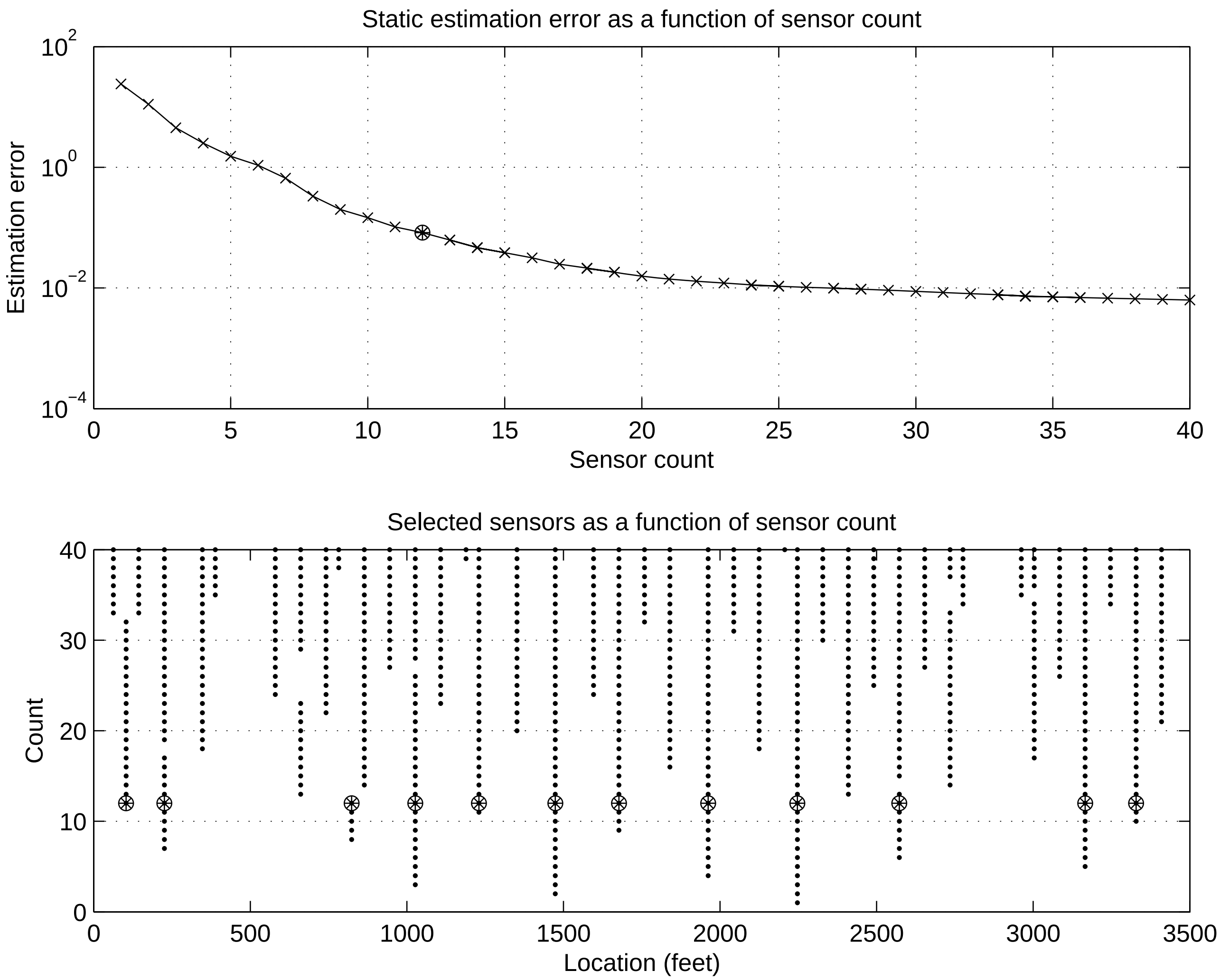

Figure 2. Sensor selection result from the static estimation method.

Approach	Time (mm:ss)
Static estimation	00:32
Observability	09:45
Dynamic estimation	56:20

Table 1. Computation time for 40 sensors.

7. CONCLUSIONS

We have proposed three approaches to Optimal Sensor Placement (OSP) for estimation-based health monitoring of civil structures: based on minimizing the static displacement estimation error, minimizing the inverse of the observability, and minimizing the dynamic displacement estimation error respectively. Each of these approaches leads to a nonlinear mixed-integer optimization problem for which a locally optimal solution can be found using an iterative search algorithm. Each of these approaches can also be relaxed to a convex optimization problem, for which a global optimal solution can be found. However, the solution to the relaxed problem must be approximated to generate a discrete sensor selection. Specialized software is required to handle the convex optimization in case of the large model orders anticipated.

Each of the approaches is implemented and applied to a simple 2D model of the long-span New Carquinez Bridge. The resulting sensor placement solutions are similar for each of these approaches, but certainly not

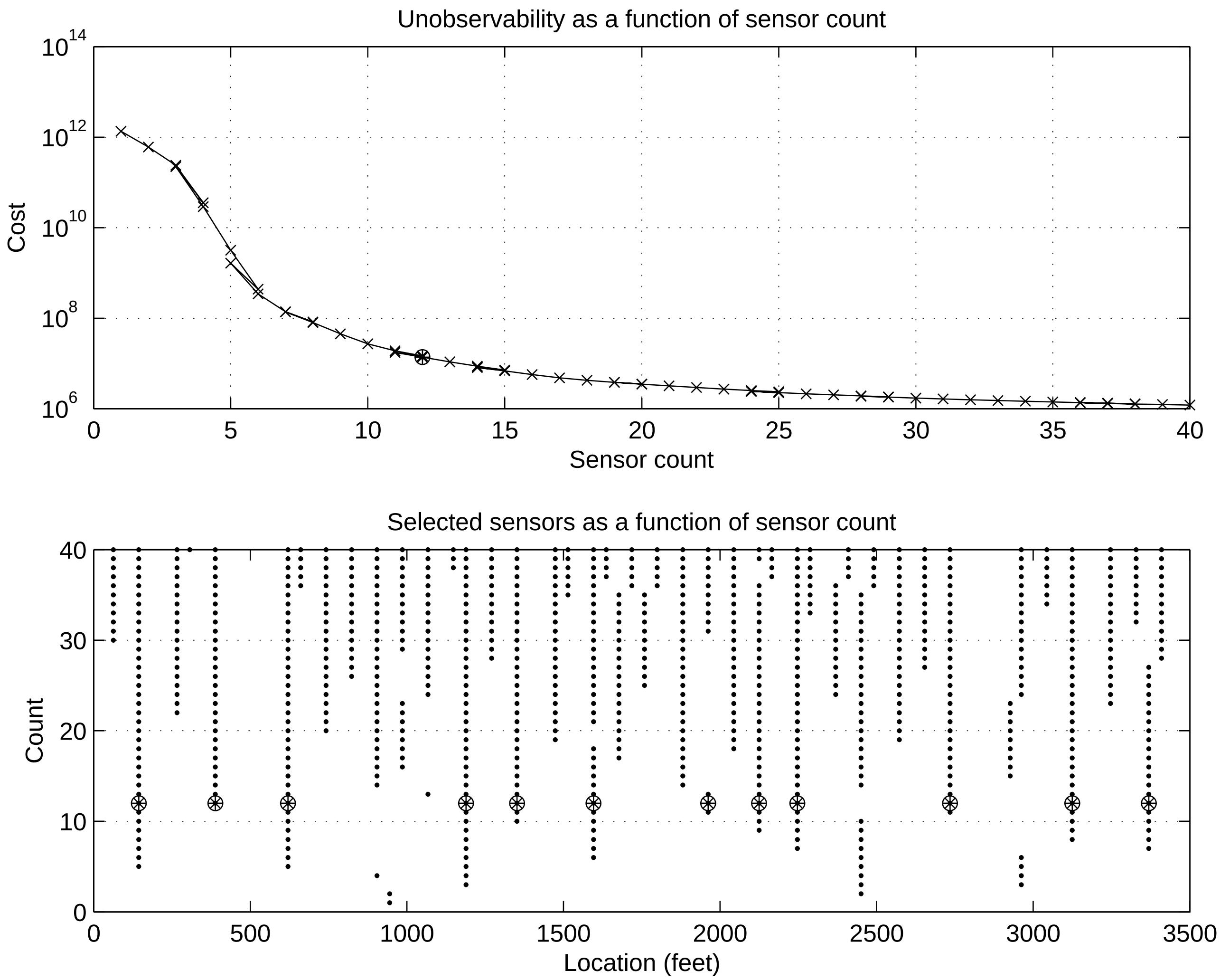

Figure 3. Sensor selection result from the observability method.

identical. While by far the most computationally intensive, the dynamic estimation-based OSP approach captures the actual sensor placement problem for state estimation-based health monitoring. More study is needed to determine if e.g. the more computationally efficient observability-based OSP approach yields sufficiently good results.

Our goal is to apply OSP to detailed finite element-based bridge models. As the next step we plan to implement the optimization using specialized efficient convex solvers, and integrate model-reduction techniques into the approach. Additionally, we plan to extend the problem to include sensors such as strain gauges and accelerometers.

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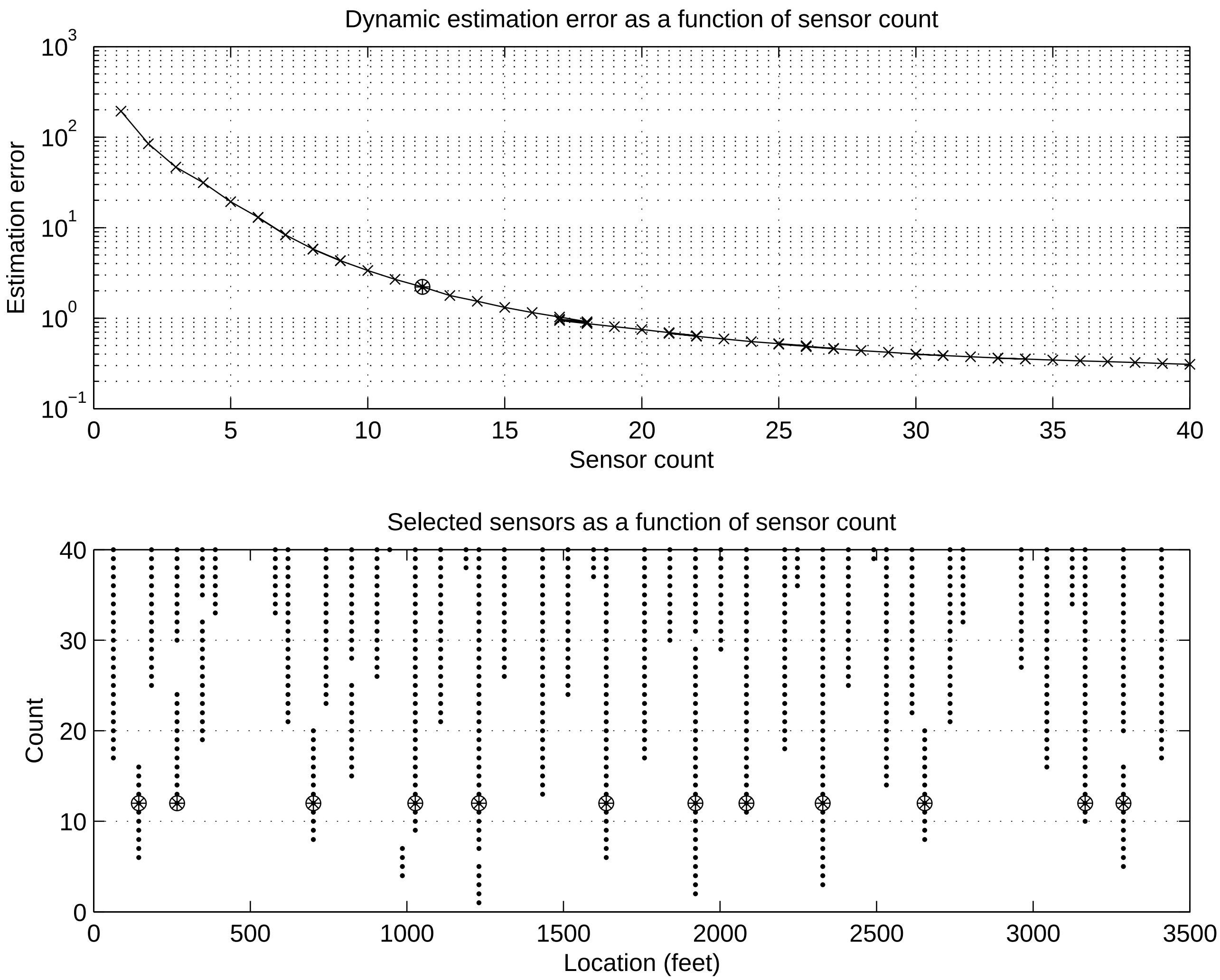

Figure 4. Sensor selection result from the dynamic estimation method.

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